

Weak convergence for a stochastic exponential integrator and finite element discretization of SPDE for multiplicative & additive noise

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Abstract

We consider a finite element approximation of a general semi-linear stochastic partial differential equation (SPDE) driven by space-time multiplicative and additive noise. We examine the full weak convergence rate of the exponential Euler scheme when the linear operator is self adjoint and provide preliminaries results toward the full weak convergence rate for non-self-adjoint linear operator. Key part of the proof does not rely on Malliavin calculus. Depending of the regularity of the noise and the initial solution, we found that in some cases the rate of weak convergence is twice the rate of the strong convergence. Our convergence rate is in agreement with some numerical results in two dimensions.

Keywords: SPDE, Finite element methods, Exponential integrators, Weak convergence, Strong convergence

1. Introduction

The weak numerical approximation of an Itô stochastic partial differential equation defined in $\Omega \subset \mathbb{R}^d$ is analyzed. Boundary conditions on the domain Ω are typically Neumann, Dirichlet or Robin conditions. More precisely, we consider in the abstract setting the following stochastic partial differential equation

$$dX = (AX + F(X))dt + B(X)dW, \quad X(0) = X_0, \quad t \in [0, T], \quad T > 0 \quad (1)$$

on the Hilbert space $L^2(\Omega)$. Here the linear operator A which is not necessarily selfadjoint, is the generator of an analytic semigroup $S(t) := e^{tA}, t \geq 0$. The functions F and B are

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nonlinear functions of X and the noise term $W(t)$ is a Q -Wiener process defined on a filtered probability space $(\mathbb{D}, \mathcal{F}, \mathbb{P}, \{F_t\}_{t \geq 0})$, that is white in time. The filtration is assumed to fulfill the usual conditions (see e.g. [25, Definition 2.1.11]). For technical reasons more interest will be on a deterministic initial value $X_0 \in H$. The noise can be represented as a series in the eigenfunctions of the covariance operator Q given by

$$W(x, t) = \sum_{i \in \mathbb{N}^d} \sqrt{q_i} e_i(x) \beta_i(t), \quad (2)$$

where (q_i, e_i) , $i \in \mathbb{N}^d$ are the eigenvalues and eigenfunctions of the covariance operator Q and β_i are independent and identically distributed standard Brownian motions. Under some technical assumptions it is well known (see [5, 25, 4]) that the unique mild solution of (1) is given by

$$X(t) = S(t)X_0 + \int_0^t S(t-s)F(X(s))ds + \int_0^t S(t-s)B(X(s))dW(s). \quad (3)$$

Equations of type (1) arise in physics, biology and engineering [27, 28, 9] and in few cases, exact solutions exist. The study of numerical solutions of SPDEs is therefore an active research area and there is an extensive literature on numerical methods for SPDE (1) [13, 14, 15, 28, 23, 1, 21, 34]. Basically there are two types of convergence. The strong convergence or pathwise convergence studies the pathwise convergence of the numerical solution to true solution while the weak convergence aims to approximate the law of the solution at a fixed time. In many applications, weak error is more relevant as interest are usually based on some functions of the solution i.e. $\mathbb{E}\Phi(X)$, where $\Phi : H \rightarrow \mathbb{R}$ and \mathbb{E} is the expectation. Strong convergence rates for numerical approximations of stochastic evolution equations of type (1) with smooth and regular nonlinearities are well understood in the scientific literature (see [13, 14, 15, 1, 21, 34, 28, 22] and references therein). Weak convergence rates for numerical approximations of equation (1) are far away from being well understood. For a linear SPDE with additive noise, the solution can be written explicitly and the weak error have been estimated in [26, 24] with implicit Euler method for time discretization. The space discretization have been performed with finite difference method [26, 24] and finite element method [24]. The weak error of the implicit Euler method is more complicated for nonlinear equation of type (1) as the Malliavin calculus is usually used to handle the irregular term and the term involving the nonlinear operators F and B (see [6, 33, 2]). In almost all the literature for weak error estimation, the linear operator A is assumed to be self adjoint. Furthermore no numerical simulations were made to sustain the theoretical results to the best of our knowledge. In this paper we consider a stochastic exponential scheme (called stochastic exponential Euler scheme) as in [22] and provide the weak error of the full discrete scheme (Theorem 6.2, Theorem 6.3 and Remark 6.3) where the space discretization is performed using finite element, following closely the works in [32, 33] on another exponential integrator scheme. Our weak convergence proof does not use Malliavin calculus. Furthermore we provide some preliminaries results (weak convergence of the semi discrete scheme in Theorem 4.1 and Theorem 5.1) toward the weak convergence when the linear operator A is not necessarily self-adjoint, and provide some numerical examples to sustain the theoretical results. Recent work in [2] is used to obtain optimal convergence order for additive noise when the linear operator is self adjoint in Theorem 6.3 and Remark 6.3. We also extend in Theorem 6.1 the strong optimal convergence rate provided in [19, Theorem

1.1] to non-self-adjoint operator A . Note that as the operator A is not necessarily self-adjoint, our scheme here are based on exponential matrix computation. The deterministic part of this scheme have be proved to be efficient and robust comparing to standard schemes in many applications [9, 28, 30, 29] where the exponential matrix functions have been computed using the Krylov subspace technique [12] and fast Leja points technique [3]. For convenience of presentation, we take A to be a second order operator as this simplifies the convergence proof. Our results can be extended to high order semi linear parabolic SPDE.

The paper is organized as follows. Section 2 provides abstract setting and the well posedness of (1). The stochastic exponential Euler scheme along with weak error representation are provided in Section 3. The temporal weak convergence rate of the stochastic exponential Euler scheme is provided in Section 4 for additive noise and in Section 5 for multiplicative noise. Note that in this section the linear operator A is not necessarily self-adjoint. Section 6 provides strong optimal convergence rate of the semi discrete solution for non-self-adjoint operator A along with full weak convergence rate of the stochastic exponential Euler scheme for self-adjoint operator A . Numerical results to sustain some theoretical results are provided in Section 7.

2. The abstract setting and mild solution

Let us start by presenting briefly the notation for the main function spaces and norms that we use in the paper. Let H be a separable Hilbert space with the norm $\|\cdot\|$ associated to the inner product $\langle \cdot, \cdot \rangle_H$. For a Banach space U we denote by $\|\cdot\|_U$ the norm of U , $L(U, H)$ the set of bounded linear mapping from U to H and by $L_2(\mathbb{D}, U)^1$ the Hilbert space of all equivalence classes of square integrable U -valued random variables. For ease of notation $L(U, U) = L(U)$. Furthermore we denote by $\mathcal{L}_1(U, H)$ the set of nuclear operators from U to H , $\mathcal{L}_2(U, H) := HS(U, H)$ the space of Hilbert Schmidt functions from U to H and $\mathcal{C}_b^k(U, H)$ the space of not neccessarily bounded mappings from U to H that have continuous and bounded Frechet derivatives up to order k , $k \in \mathbb{N}$. For simplicity we also write $\mathcal{L}_1(U, U) = \mathcal{L}_1(U)$ and $\mathcal{L}_2(U, U) = \mathcal{L}_2(U)$.

For a given orthonormal basis (e_i) of U , the trace of $l \in \mathcal{L}_1(U)$ is defined by

$$\text{Tr}(l) := \sum_{i \in \mathbb{N}^d} \langle l e_i, e_i \rangle_U, \quad (4)$$

while the norm of $l \in \mathcal{L}_2(U)$ is defined by

$$\|l\|_{\mathcal{L}_2(U)}^2 := \sum_{i \in \mathbb{N}^d} \|l e_i\|_U^2 < \infty, \quad (5)$$

Note that the trace in (4) and the Hilbert Schmidt norm in (5) are independent of the basis (e_i) .

¹ \mathbb{D} is the sample space

Let $Q : H \rightarrow H$ be an operator, we consider throughout this work the Q -Wiener process. We denote the space of Hilbert–Schmidt operators from $Q^{1/2}(H)$ to H by $L_2^0 := \mathcal{L}_2(Q^{1/2}(H), H) = HS(Q^{1/2}(H), H)$ and the corresponding norm $\|\cdot\|_{L_2^0}$ by

$$\|l\|_{L_2^0} := \|lQ^{1/2}\|_{\mathcal{L}_2(H)} = \left(\sum_{i \in \mathbb{N}^d} \|lQ^{1/2}e_i\|^2 \right)^{1/2}, \quad l \in L_2^0.$$

Let $\varphi : [0, T] \times \Omega \rightarrow L_2^0$ be a L_2^0 -valued predictable stochastic process with $\mathbb{P} \left[\int_0^T \|\varphi\|_{L_2^0}^2 ds < \infty \right] = 1$. Then Ito's isometry (see e.g. [5, Step 2 in Section 2.3.2]) gives

$$\mathbb{E} \left\| \int_0^t \varphi dW \right\|^2 = \int_0^t \mathbb{E} \|\varphi\|_{L_2^0}^2 ds = \int_0^t \mathbb{E} \|\varphi Q^{1/2}\|_{\mathcal{L}_2(H)}^2 ds, \quad t \in [0, T].$$

Let us recall the following proprieties which will be used in our errors estimation.

Proposition 2.1. [4] *Let l, l_1, l_2 be three operators in Banach spaces, the following proprieties hold*

- If $l \in \mathcal{L}_1(U)$ then

$$|\text{Tr}(l)| \leq \|l\|_{\mathcal{L}_1(U)}. \quad (6)$$

- If $l_1 \in L(H)$ and $l_2 \in \mathcal{L}_1(H)$, then both $l_1 l_2$ and $l_2 l_1$ belong to $\mathcal{L}_1(H)$ with

$$\text{Tr}(l_1 l_2) = \text{Tr}(l_2 l_1). \quad (7)$$

- If $l_1 \in \mathcal{L}_2(U, H)$ and $l_2 \in \mathcal{L}_2(H, U)$, then $l_1 l_2 \in \mathcal{L}_1(H)$ with

$$\|l_1 l_2\|_{\mathcal{L}_1(H)} \leq \|l_1\|_{\mathcal{L}_2(U, H)} \|l_2\|_{\mathcal{L}_2(H, U)}. \quad (8)$$

- If $l \in \mathcal{L}_2(U, H)$, then its adjoint $l^* \in \mathcal{L}_2(H, U)$ with

$$\|l^*\|_{\mathcal{L}_2(H, U)} = \|l\|_{\mathcal{L}_2(U, H)}. \quad (9)$$

- If $l \in L(U, H)$ and $l_j \in \mathcal{L}_j(U)$, $j = 1, 2$, then $ll_j \in \mathcal{L}_j(U, H)$ with

$$\|ll_j\|_{\mathcal{L}_j(U, H)} \leq \|l\|_{L(U, H)} \|l_j\|_{\mathcal{L}_j(U)}, \quad j = 1, 2. \quad (10)$$

For classical well posedness, some assumptions are required both for the existence and uniqueness of the solution of equation (1).

Assumption 2.1. *The operator $A : \mathcal{D}(A) \subset H \rightarrow H$ is a negative generator of an analytic semigroup $S(t) = e^{tA}$, $t \geq 0$.*

In the Banach space $\mathcal{D}((-A)^{\alpha/2})$, $\alpha \in \mathbb{R}$, we use the notation $\|(-A)^{\alpha/2}\cdot\| =: \|\cdot\|_{\alpha}$. We recall some basic properties of the semigroup $S(t)$ generated by A .

Proposition 2.2. [Smoothing properties of the semigroup [11]]

Let $\alpha > 0$, $\beta \geq 0$ and $0 \leq \gamma \leq 1$, then there exist $C > 0$ such that

$$\begin{aligned} \|(-A)^\beta S(t)\|_{L(H)} &\leq Ct^{-\beta} \quad \text{for } t > 0 \\ \|(-A)^{-\gamma}(I - S(t))\|_{L(H)} &\leq Ct^\gamma \quad \text{for } t \geq 0. \end{aligned}$$

In addition,

$$\begin{aligned} (-A)^\beta S(t) &= S(t)(-A)^\beta \quad \text{on } \mathcal{D}((-A)^\beta) \\ \text{If } \beta &\geq \gamma \quad \text{then } \mathcal{D}((-A)^\beta) \subset \mathcal{D}((-A)^\gamma), \\ \|D_t^l S(t)v\|_\beta &\leq Ct^{-l-(\beta-\alpha)/2} \|v\|_\alpha, \quad t > 0, v \in \mathcal{D}((-A)^{\alpha/2}) \quad l = 0, 1, \end{aligned}$$

where $D_t^l := \frac{d^l}{dt^l}$.

We describe now in detail the standard assumptions usually used on the nonlinear terms F, B and the noise W .

Assumption 2.2. [Assumption on the drift term F] There exists a positive constant $L > 0$ such that $F : H \rightarrow H$ satisfies the following Lipschitz condition

$$\|F(Z) - F(Y)\| \leq L\|Z - Y\| \quad \forall Z, Y \in H.$$

As a consequence, there exists a constant $C > 0$ such that

$$\|F(Z)\| \leq \|F(0)\| + \|F(Z) - F(0)\| \leq \|F(0)\| + L\|Z\| \leq C(1 + \|Z\|) \quad Z \in H.$$

Assumption 2.3. [Assumption on the diffusion term B]

There exists a positive constant $L > 0$ such that the mapping $B : H \rightarrow \mathcal{L}_2(H)$ satisfies the following condition

$$\|B(Z) - B(Y)\|_{\mathcal{L}_2(H)} \leq L\|Z - Y\| \quad \forall Z, Y \in H.$$

As a consequence, there exists a constant $C > 0$ such that

$$\begin{aligned} \|B(Z)\|_{\mathcal{L}_2(H)} &\leq \|B(0)\|_{\mathcal{L}_2(H)} + \|B(Z) - B(0)\|_{\mathcal{L}_2(H)} \\ &\leq \|B(0)\|_{\mathcal{L}_2(H)} + L\|Z\| \leq C(1 + \|Z\|) \quad Z \in H. \end{aligned} \quad (11)$$

Theorem 2.1. [Existence and uniqueness ([5])]

Assume that the initial solution X_0 is an F_0 -measurable H -valued random variable and Assumption 2.1, Assumption 2.2, Assumption 2.3 are satisfied. There exists a mild solution X to (1) unique, up to equivalence among the processes, satisfying

$$\mathbb{P} \left(\int_0^T \|X(s)\|^2 ds < \infty \right) = 1. \quad (12)$$

For any $p \geq 2$ there exists a constant $C = C(p, T) > 0$ such that

$$\sup_{t \in [0, T]} \mathbb{E} \|X(t)\|^p \leq C (1 + \mathbb{E} \|X_0\|^p). \quad (13)$$

For any $p > 2$ there exists a constant $C_1 = C_1(p, T) > 0$ such that

$$\mathbb{E} \sup_{t \in [0, T]} \|X(t)\|^p \leq C_1 (1 + \mathbb{E} \|X_0\|^p). \quad (14)$$

The following theorem proves a regularity result of the mild solution X of (1).

Theorem 2.2. [Regularity of the mild solution ([22])]

Assume that Assumption 2.1, Assumption 2.2 and Assumption 2.3 hold. Let X be the mild solution of (1) given in (3). If $X_0 \in L_2(\mathbb{D}, \mathcal{D}((-A)^{\beta/2}))$, $\beta \in [0, 1)$ then for all $t \in [0, T]$, $X(t) \in L_2(\mathbb{D}, \mathcal{D}((-A)^{\beta/2}))$ with

$$(\mathbb{E} \|X(t)\|_{\beta}^2)^{1/2} \leq C \left(1 + (\mathbb{E} \|X_0\|_{\beta}^2)^{1/2}\right).$$

For the weak error representation, we will need the following lemma.

Lemma 2.1 (Itô's formula). Let $(\mathbb{D}, \mathcal{F}, P; \{F_t\}_{t \geq 0})$ be a filtered probability space. Let ϕ and Ψ be H -valued predictable processes, Bochner integrable on $[0, T]$ P -almost surely (see [17]), and Y_0 be an F_0 -measurable H -valued random variable. Let $G : [0, T] \times H \rightarrow \mathbb{R}$ and assume that the Fréchet derivatives $G_t(t, x)$, $G_x(t, x)$, and $G_{xx}(t, x)$ are uniformly continuous as functions of (t, x) on bounded subsets of $[0, T] \times H$. Note that, for fixed t , $G_x(t, x) \in L(H, \mathbb{R})$ and we consider $G_{xx}(t, x)$ as an element of $L(H)$. Let W be the Q -Wiener process. If Y satisfies

$$Y(t) = Y(0) + \int_0^t \phi(s) ds + \int_0^t \Psi(s) dW(s), \quad (15)$$

then P -almost surely for all $t \in [0, T]$,

$$\begin{aligned} G(t, Y(t)) &= G(0, Y(0)) + \int_0^t G_x(s, Y(s)) \Psi(s) dW(s) \\ &\quad + \int_0^t \left\{ G_t(s, Y(s)) + G_x(s, Y(s)) \phi(s) + \frac{1}{2} \text{Tr} \left(G_{xx}(s, Y(s)) \Psi(s) Q^{\frac{1}{2}} (\Psi(s) Q^{\frac{1}{2}})^* \right) \right\} ds \end{aligned} \quad (16)$$

A proof of this lemma can be found in [5].

3. Application to the second order semi-linear parabolic SPDE

We assume that Ω has a smooth boundary or is a convex polygon of \mathbb{R}^d , $d = 1, 2, 3$. In the sequel, for convenience of presentation, we take A to be a second order operator as this simplifies the convergence proof. The result can be extended to high order semi linear parabolic SPDE.

More precisely we take $H = L^2(\Omega)$ and consider the general second order semi-linear parabolic stochastic partial differential equation given by

$$dX(t, x) = (\nabla \cdot \mathbf{D} \nabla X(t, x) - \mathbf{q} \cdot \nabla X(t, x) + f(x, X(t, x))) dt + b(x, X(t, x)) dW(t, x), \quad (17)$$

$x \in \Omega, t \in [0, T]$ where $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a globally Lipschitz continuous function and $b : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuously differentiable function with globally bounded derivatives.

3.1. The abstract setting for second order semi-linear parabolic SPDE

In the abstract form given in (1), the nonlinear functions $F : H \rightarrow H$ and $B : H \rightarrow HS(Q^{1/2}(H), H)$ are defined by

$$(F(v))(x) = f(x, v(x)), \quad (B(v)u)(x) = b(x, v(x)) \cdot u(x), \quad (18)$$

for all $x \in \Omega, v \in H, u \in Q^{1/2}(H)$, with $H = L^2(\Omega)$. Note that we can also define $B : H \rightarrow \mathcal{L}_2(H)$ by

$$(B(v)u)(x) = b(x, v(x)) \cdot Q^{1/2}u(x), \quad (19)$$

for all $x \in \Omega, v \in H, u \in H$.

In order to define rigorously the linear operator, let us set

$$\mathcal{A} = \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left(D_{i,j} \frac{\partial}{\partial x_j} \right) - \sum_{i=1}^d q_i \frac{\partial}{\partial x_i}, \quad (20)$$

where we assume that $D_{i,j} \in L^\infty(\Omega), q_i \in L^\infty(\Omega)$ and that there exists a positive constant $c_1 > 0$ such that

$$\sum_{i,j=1}^d D_{i,j}(x) \xi_i \xi_j \geq c_1 |\xi|^2, \quad \forall \xi \in \mathbb{R}^d, \quad x \in \overline{\Omega}, \quad c_1 > 0. \quad (21)$$

We introduce two spaces \mathbb{H} and V where $\mathbb{H} \subset V$ depends on the choice of the boundary conditions for the SPDE. For Dirichlet boundary conditions we let

$$V = \mathbb{H} = H_0^1(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\},$$

and for Robin boundary conditions, Neumann boundary being a special case, we take $V = H^1(\Omega)$ and

$$\mathbb{H} = \{v \in H^2(\Omega) : \partial v / \partial \nu_{\mathcal{A}} + \alpha_0 v = 0 \text{ on } \partial\Omega\}, \quad \alpha_0 \in \mathbb{R}.$$

See [8] for details. The corresponding bilinear form of $-\mathcal{A}$ is given by

$$a(u, v) = \int_{\Omega} \left(\sum_{i,j=1}^d D_{i,j} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + \sum_{i=1}^d q_i \frac{\partial u}{\partial x_j} v \right) dx, \quad u, v \in V \quad (22)$$

for Dirichlet and Neumann boundary conditions, and by

$$a(u, v) = \int_{\Omega} \left(\sum_{i,j=1}^d D_{i,j} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + \sum_{i=1}^d q_i \frac{\partial u}{\partial x_j} v \right) dx + \int_{\partial\Omega} \alpha_0 u v dx, \quad u, v \in V, \quad (23)$$

for Robin boundary conditions. According to Gårding's inequality (see [28, 8]), there exist two positive constants c_0 and λ_0 such that

$$a(v, v) + c_0 \|v\|^2 \geq \lambda_0 \|v\|_{H^1(\Omega)}^2, \quad \forall v \in V. \quad (24)$$

By adding and subtracting $c_0 X dt$ on the right hand side of (17), we have a new operator that we still call \mathcal{A} corresponding to the new bilinear form that we still call a such that the following coercivity property holds

$$a(v, v) \geq \lambda_0 \|v\|_{H^1(\Omega)}^2, \quad \forall v \in V. \quad (25)$$

Note that the expression of the nonlinear term F has changed as we include the term $-c_0 X$ in a new nonlinear term that we still denote by F .

Note that $a(\cdot, \cdot)$ is bounded in $V \times V$, so the following operator $A : V \rightarrow V^*$ is well defined by the Riez's representation theorem

$$a(u, v) = -\langle Au, v \rangle, \quad \forall u, v \in V, \quad (26)$$

where V^* is the adjoint space (or dual space) of V and $\langle \cdot, \cdot \rangle$ the duality pairing between V^* and V . By identifying H to its adjoint space H^* , we get the following continuous and dense inclusions

$$V \subset H \subset V^*. \quad (27)$$

So, we have

$$\langle u, v \rangle_H = \langle u, v \rangle \quad \forall u \in H, \forall v \in V. \quad (28)$$

The linear operator A in our abstract setting (1) is therefore defined by (26). The domain of A denoted by $\mathcal{D}(A)$ is defined by

$$\mathcal{D}(A) = \{u \in V, Au \in H\}. \quad (29)$$

We write the restriction of $A : V \rightarrow V^*$ to $\mathcal{D}(A)$ again by A , which is therefore regarded as an operator of H (more precisely the H realization of \mathcal{A} [8, p. 812]). The coercivity property (25) implies that A is a sectorial on $L^2(\Omega)$ i.e. there exists C_1 , $\theta \in (\frac{1}{2}\pi, \pi)$ such that

$$\|(\lambda I - A)^{-1}\|_{L(L^2(\Omega))} \leq \frac{C_1}{|\lambda|} \quad \lambda \in S_{\theta}, \quad (30)$$

where $S_{\theta} = \{\lambda \in \mathbb{C} : \lambda = \rho e^{i\phi}, \rho > 0, 0 \leq |\phi| \leq \theta\}$ (see [11, 8]). Then A is the infinitesimal generator of bounded analytic semigroups $S(t) := e^{tA}$ on $L^2(\Omega)$ such that

$$S(t) := e^{tA} = \frac{1}{2\pi i} \int_{\mathcal{C}} e^{t\lambda} (\lambda I - A)^{-1} d\lambda, \quad t > 0, \quad (31)$$

where \mathcal{C} denotes a path that surrounds the spectrum of A . The condition property (25) also implied that $-A$ is a positive operator and its fractional powers is well defined for any $\alpha > 0$, by

$$\begin{cases} (-A)^{-\alpha} &= \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{tA} dt, \\ (-A)^\alpha &= ((-A)^{-1})^{-1}, \end{cases} \quad (32)$$

where $\Gamma(\alpha)$ is the Gamma function (see [11]). Functions in \mathbb{H} satisfy the boundary conditions and with \mathbb{H} in hand we can characterize the domain of the operator $(-A)^{\alpha/2}$ and have the following norm equivalence ([8, 20, 7]) for $\alpha \in \{1, 2\}$

$$\begin{aligned} \|v\|_{H^\alpha(\Omega)} &\equiv \|(-A)^{\alpha/2}v\| =: \|v\|_\alpha, & \forall v \in \mathcal{D}((-A)^{\alpha/2}), \\ \mathcal{D}((-A)^{\alpha/2}) &= \mathbb{H} \cap H^r(\Omega) & (\text{Dirichlet boundary conditions}), \\ \mathcal{D}(-A) &= \mathbb{H}, & \mathcal{D}((-A)^{1/2}) = H^1(\Omega) \quad (\text{Robin boundary conditions}). \end{aligned}$$

3.2. Numerical scheme and weak error representation

We consider the discretization of the spatial domain by a finite element method. Let \mathcal{T}_h be a set of disjoint intervals of Ω (for $d = 1$), a triangulation of Ω (for $d = 2$) or a set of tetrahedra (for $d = 3$) with maximal length h satisfying the usual regularity assumptions [16, 10].

Let $V_h \subset V$ denote the space of continuous functions that are piecewise linear over the triangulation \mathcal{T}_h . Note that for high order polynomial, high order accuracy in space can be achieved. To discretize in space we introduce the projection P_h from $L^2(\Omega)$ onto V_h defined for $u \in L^2(\Omega)$ by

$$(P_h u, \chi) = (u, \chi), \quad \forall \chi \in V_h. \quad (33)$$

The discrete operator $A_h : V_h \rightarrow V_h$ is defined by

$$(A_h \varphi, \chi) = (A \varphi, \chi) = -a(\varphi, \chi), \quad \varphi, \chi \in V_h. \quad (34)$$

Like the operator A , the discrete operator A_h is also the generator of an analytic semigroup $S_h := e^{tA_h}$. Here we consider the following semi-discrete form of the problem (1), which consists to find the process $X^h(t) = X^h(\cdot, t) \in V_h$ such that for $t \in [0, T]$,

$$dX^h = (A_h X^h + P_h F(X^h))dt + P_h B(X^h)P_h dW, \quad X^h(0) =: X_0^h = P_h X_0. \quad (35)$$

Note that (35) is a finite dimensional stochastic equation. The mild solution of (35) at time $t_m = m\Delta t$ is given by

$$\begin{aligned} X^h(t_m) &= S_h(t_m)P_h X_0 + \int_0^{t_m} S_h(t_m - s)P_h F(X^h(s))ds \\ &\quad + \int_0^{t_m} S_h(t_m - s)P_h B(X^h(s))P_h dW(s), \end{aligned} \quad (36)$$

where $\Delta t = T/M$, $m \in \{0, 1, \dots, M\}$, $M \in \mathbb{N}$.

Then, given the mild solution at the time t_m , we can construct the corresponding solution at t_{m+1} as

$$\begin{aligned} X^h(t_{m+1}) &= S_h(\Delta t)X^h(t_m) + \int_0^{\Delta t} S_h(\Delta t - s)P_h F(X^h(s + t_m))ds \\ &\quad + \int_{t_m}^{t_{m+1}} S_h(t_{m+1} - s)P_h B(X^h(s))P_h dW(s). \end{aligned}$$

To build the numerical scheme, we use the following approximations [22]

$$\begin{aligned} P_h F(X^h(t_m + s)) &\approx P_h F(X^h(t_m)), \quad s \in [0, \Delta t], \\ S_h(t_{m+1} - s)P_h B(X^h(s)) &\approx S_h(\Delta t)P_h B(X^h(t_m)), \quad s \in [t_m, t_{m+1}]. \end{aligned}$$

We can define our approximation X_m^h of $X(m\Delta t)$ by

$$X_{m+1}^h = e^{\Delta t A_h} X_m^h + A_h^{-1} (e^{\Delta t A_h} - I) P_h F(X_m^h) + e^{\Delta t A_h} P_h B(X_m^h) P_h (W_{t_{m+1}} - W_{t_m}). \quad (37)$$

We introduce for brevity the notations

$$S_h(t) = e^{tA_h}, \quad S_h^1(t) = (tA_h)^{-1}(e^{tA_h} - I), \quad B_h : X \mapsto P_h B(X)P_h.$$

We can then rewrite the scheme (37) as

$$X_{m+1}^h = S_h(\Delta t)X_m^h + \Delta t S_h^1(\Delta t)P_h F(X_m^h) + S_h(\Delta t)B_h(X_m^h) (W_{t_{m+1}} - W_{t_m}). \quad (38)$$

In order to study the weak convergence of the approximation of the solutions we define the functional

$$\mu^h(t, \psi) = \mathbb{E} [\Phi(X^h(t, \psi))], \quad t \in [0, T], \quad \psi \in V_h, \quad (39)$$

where $X^h(t, \psi)$ is defined by (36) with the initial value $X_0^h = \psi \in V_h$. It can be shown (see Theorem 9.16 of [9]) that $\mu^h(t, \psi)$ defined by (39) is differentiable with respect to t and twice differentiable with respect to ψ , and is the unique strict solution of

$$\begin{cases} \frac{\partial \mu^h}{\partial t}(t, \psi) &= \langle A_h \psi + P_h F(\psi), D\mu^h(t, \psi) \rangle_H + \frac{1}{2} \text{Tr} [D^2 \mu^h(t, \psi) B_h(\psi) B_h(\psi)^*] \\ \mu^h(0, \psi) &= \Phi(\psi), \quad \psi \in V_h. \end{cases} \quad (40)$$

We can here, using the Riesz representation theorem, identify the first order derivative of $\mu^h(t, \psi)$ with respect to ψ , denoted as $D\mu^h(t, \psi)$ with an element of V_h and the second derivative denoted as $D^2 \mu^h(t, \psi)$ with a linear operator in V_h . More precisely

$$D\mu^h(t, \psi)(\phi_1) = \langle D\mu^h(t, \psi), \phi_1 \rangle_H, \quad \forall \psi, \phi_1 \in V_h, \quad (41)$$

$$D^2 \mu^h(t, \psi)(\phi_1, \phi_2) = \langle D^2 \mu^h(t, \psi) \phi_1, \phi_2 \rangle_H, \quad \forall \psi, \phi_1, \phi_2 \in V_h. \quad (42)$$

The following theorem, which is similar to [32, Theorem 2.2] is fundamental for our convergence proofs.

Theorem 3.1 (Weak error representation formula for the semi discrete problem).

Assume that all the conditions in the assumptions above are fulfilled and let $\{W(t)\}_{t \in [0, T]}$ be a cylindrical H -valued Q -Wiener process. Then for $\Phi \in \mathcal{C}_b^2(H, \mathbb{R})$ the weak approximation error of the scheme in (38) has the representation

$$\begin{aligned} & \mathbb{E}[\Phi(X_M^h)] - \mathbb{E}[\Phi(X^h(T))] \\ &= \sum_{m=0}^{M-1} \left\{ \int_{t_m}^{t_{m+1}} \mathbb{E} \left[\langle D\mu^h(T-s, \tilde{X}^h(s)), P_h F(X_m^h) - P_h F(\tilde{X}^h(s)) \rangle_H \right] ds \right. \\ & \quad \left. + \frac{1}{2} \mathbb{E} \int_{t_m}^{t_{m+1}} \text{Tr} \left[D^2 \mu^h(T-s, \tilde{X}^h(s)) \left(\{S_h(s-t_m)B_h(X_m^h)\} \{S_h(s-t_m)B_h(X_m^h)\}^* \right. \right. \right. \\ & \quad \left. \left. \left. - B_h(X_m^h)B_h(X_m^h)^* \right) \right] ds \right\}, \end{aligned} \quad (43)$$

where $\tilde{X}^h(t)$ is a continuous extension of X_m^h , defined by

$$\begin{aligned} \tilde{X}^h(t) &= S_h(t-t_m)X_m^h + (t-t_m)S_h^1(t-t_m)P_h F(X_m^h) \\ & \quad + S_h(t-t_m)B_h(X_m^h)(W(t) - W(t_m)), \quad \text{for } t \in [t_m, t_{m+1}]. \end{aligned} \quad (44)$$

Proof. Introduce the process $\nu^h : [0, T] \times V_h \rightarrow \mathbb{R}$, given by

$$\nu^h(x, \psi) = \mu^h(t, S_h(-t)\psi), \quad (45)$$

which is twice differentiable with respect to ψ and by previous identifications satisfies

$$D\nu^h(t, \psi)\phi = \langle D\mu^h(t, S_h(-t)\psi), S_h(-t)\phi \rangle_H, \quad \psi, \phi \in V_h \quad (46)$$

$$D^2\nu^h(t, \psi)(\phi_1, \phi_2) = \langle D^2\mu^h(t, S_h(-t)\psi)(S_h(-t)\phi_1), S_h(-t)\phi_2 \rangle_H, \quad \psi, \phi_1, \phi_2 \in V_h. \quad (47)$$

One can then check that $\nu^h(t, \psi)$ solves the equation

$$\begin{aligned} \frac{\partial \nu^h}{\partial t}(t, \psi) &= \langle D\nu^h(t, \psi), S_h(t)P_h F(S_h(-t)\psi) \rangle_H + \frac{1}{2} \text{Tr} [D^2\nu^h(t, \psi)S_h(t)B_h(\psi)(S_h(t)B_h(\psi))^*], \\ \nu^h(0, \psi) &= \Phi(\psi), \quad \psi \in V_h. \end{aligned} \quad (48)$$

Indeed, since $\frac{\partial S_h(-t)\psi}{\partial t} = -A_h S_h(-t)\psi$, we have that

$$\frac{\partial \nu^h}{\partial t}(t, \psi) = \frac{\partial \mu^h}{\partial t}(t, S_h(-t)\psi) + \langle D\mu^h(t, S_h(-t)\psi), -A_h S_h(-t)\psi \rangle_H.$$

Using the Kolmogorov equation (40) and the fact that $\langle \cdot, \cdot \rangle_H$ is symmetric yields

$$\begin{aligned} \frac{\partial \nu^h}{\partial t}(t, \psi) &= \langle A_h S_h(-t)\psi + P_h F(S_h(-t)\psi), D\mu^h(t, S_h(-t)\psi) \rangle_H \\ & \quad + \frac{1}{2} \text{Tr} [D^2\mu^h(t, S_h(-t)\psi)S_h(-t)S_h(-t)S_h(t)S_h(t)B_h(\psi)^*] \\ & \quad + \langle D\mu^h(t, S_h(-t)\psi), -A_h S_h(-t)\psi \rangle_H. \end{aligned}$$

As $D^2\mu^h(t, \psi)$ is identified as linear operator in V_h , using Proposition 2.1 (mainly relation (7)), the definition of the trace (4), the expression (47) and the fact that $\langle \cdot, \cdot \rangle_H$ is symmetric allow to have

$$\begin{aligned} \frac{\partial \nu^h}{\partial t}(t, \psi) &= \langle D\mu^h(t, S_h(-t)\psi)S_h(-t)\psi, S_h(t)P_h F(S_h(-t)\psi) \rangle_H \\ & \quad + \frac{1}{2} \text{Tr} [D^2\nu^h(t, \psi)S_h(t)B_h(\psi)(S_h(t)B_h(\psi))^*] \\ &= \langle D\nu^h(t, \psi), S_h(t)P_h F(S_h(-t)\psi) \rangle_H + \frac{1}{2} \text{Tr} [D^2\nu^h(t, \psi)S_h(t)B_h(\psi)(S_h(t)B_h(\psi))^*]. \end{aligned}$$

Now let $Z^h(t) = S_h(T-t)X^h(t)$, using (36) and the notation $X_0^h = P_h X_0$, we get

$$\begin{aligned} Z^h(t) &= S_h(T)X_0^h + \int_0^t S_h(T-s)P_h F(X^h(s))ds \\ &\quad + \int_0^t S_h(T-s)B_h(X^h(s))dW(s), \quad t \in [0, T]. \end{aligned} \quad (49)$$

Also let $\tilde{Z}^h(t) = S_h(T-t)\tilde{X}^h(t)$, using (44), we get

$$\begin{aligned} \tilde{Z}^h(t) &= S_h(T-t_m)X_m^h + (t-t_m)S_h(T-t)S_h^1(t-t_m)P_h F(X_m^h) \\ &\quad + S_h(T-t_m)B_h(X_m^h)(W(t) - W(t_m)), \quad \text{for } t \in [t_m, t_{m+1}]. \end{aligned} \quad (50)$$

which can also be written as

$$\begin{aligned} \tilde{Z}^h(t) &= S_h(T-t_m)X_m^h + \int_{t_m}^t S_h(T-s)P_h F(X_m^h)ds \\ &\quad + \int_{t_m}^t S_h(T-t_m)B_h(X_m^h)dW(s), \quad \text{for } t \in [t_m, t_{m+1}]. \end{aligned} \quad (51)$$

Consequently, we obtain

$$\tilde{Z}^h(T) = \tilde{X}^h(t) = X_M^h, \quad Z^h(0) = S_h(T)\tilde{X}^h(0) = S_h(T)X_0^h.$$

Now the weak error is worked out as

$$\begin{aligned} \mathbb{E}[\Phi(X_M^h)] - \mathbb{E}[\Phi(X^h(T, X_0^h))] &= \mathbb{E}[\Phi(\tilde{Z}^h(T))] - \mathbb{E}[\mu^h(T, X_0^h)] \\ &= \mathbb{E}[\nu^h(0, \tilde{Z}^h(T))] - \mathbb{E}[\nu^h(T, \tilde{Z}^h(0))] \\ &= \sum_{m=0}^{M-1} \mathbb{E}[\nu^h(T-t_{m+1}, \tilde{Z}^h(t_{m+1}))] - \mathbb{E}[\nu^h(T-t_m, \tilde{Z}^h(t_m))]. \end{aligned}$$

Now using Itô formula (see Lemma 2.1) applied to $G(t, x) = \nu^h(T-t, \tilde{Z}^h(t))$ in the interval $[t_m, t_{m+1}]$ and the fact that the Itô integral vanishes, we can write

$$\begin{aligned} &\mathbb{E}[\nu^h(T-t_{m+1}, \tilde{Z}^h(t_{m+1}))] - \mathbb{E}[\nu^h(T-t_m, \tilde{Z}^h(t_m))] \\ &= - \int_{t_m}^{t_{m+1}} \mathbb{E} \left[\frac{\partial \nu^h}{\partial t}(T-s, \tilde{Z}^h(s)) \right] ds + \int_{t_m}^{t_{m+1}} \mathbb{E} \left[D\nu^h(T-s, \tilde{Z}^h(s)) S_h(T-s) P_h F(X_m^h) \right] ds \\ &\quad + \frac{1}{2} \mathbb{E} \int_{t_m}^{t_{m+1}} \text{Tr} \left[D^2 \nu^h(T-s, \tilde{Z}^h(s)) \{S_h(T-t_m)B_h(X_m^h)\} \{S_h(T-t_m)B_h(X_m^h)\}^* \right] ds. \end{aligned}$$

Using the fact that $D\nu^h(t, \psi)$ is identified to an element of V_h (see the analogue representation at (41)) and (48), we finally have

$$\begin{aligned}
& \mathbb{E}[\nu^h(T - t_{m+1}, \tilde{Z}^h(t_{m+1}))] - \mathbb{E}[\nu^h(T - t_m, \tilde{Z}^h(t_m))] \\
&= \int_{t_m}^{t_{m+1}} \mathbb{E} \left[\langle D\nu^h(T - s, \tilde{Z}^h(s)), -S_h(T - s)P_h F(S_h(s - T)\tilde{Z}^h(s)) + S_h(T - s)P_h F(X_m^h) \rangle_H \right] ds \\
&\quad + \frac{1}{2} \mathbb{E} \left\{ \int_{t_m}^{t_{m+1}} \text{Tr} \left[D^2\nu^h(T - s, \tilde{Z}^h(s)) \left(\{S_h(T - t_m)B_h(X_m^h)\} \{S_h(T - t_m)B_h(X_m^h)\}^* \right. \right. \right. \\
&\quad \left. \left. \left. - S_h(T - s)B_h(X_m^h)(S_h(T - s)B_h(X_m^h))^* \right) \right] ds \right\} \\
&= \int_{t_m}^{t_{m+1}} \mathbb{E} \left[\langle D\mu^h(T - s, \tilde{X}^h(s)), P_h F(X_m^h) - P_h F(\tilde{X}^h(s)) \rangle_H \right] ds \\
&\quad + \frac{1}{2} \mathbb{E} \left\{ \int_{t_m}^{t_{m+1}} \text{Tr} \left[D^2\mu^h(T - s, \tilde{X}^h(s)) \left(\{S_h(s - t_m)B_h(X_m^h)\} \{S_h(s - t_m)B_h(X_m^h)\}^* \right. \right. \right. \\
&\quad \left. \left. \left. - B_h(X_m^h)B_h(X_m^h)^* \right) \right] ds \right\}.
\end{aligned}$$

■

4. Weak convergence for a SPDE with additive noise

In this section, we consider the additive noise where $B = Q^{1/2}$. In order to prove our weak error estimate the following weak assumptions² [32] will be used.

Assumption 4.1. [*Assumption on nonlinear function F , and Q*] We assume that $F : H \rightarrow H$ is Lipschitz and twice continuously differentiable and satisfies

$$\|F(X)\| \leq L(1 + \|X\|), \quad X \in H, \quad (52)$$

$$\|F'(Z)(X)\| \leq L\|X\|, \quad X, Z \in H, \quad (53)$$

$$\|F''(Z)(X_1, X_2)\| \leq L\|X_1\|\|X_2\|, \quad Z, X, X_1, X_2 \in H, \quad (54)$$

$$\|(-A)^{-\beta} F''(Z)(X_1, X_2)\| \leq L\|X_1\|\|X_2\|, \quad Z, X_1, X_2 \in H, \text{ for some } \beta \in [0, 1], \quad (55)$$

$$\|(-A)^{-\frac{\gamma}{2}} F'(Z)(X)\| \leq L(1 + \|Z\|_1)\|X\|_{-1}, \quad (56)$$

$$X \in H, \quad Z \in \mathcal{D}((-A)^{1/2}) \text{ for some } \gamma \in [1, 2).$$

Furthermore, we assume that the covariance operator Q satisfies

$$\|(-A)^{\frac{\beta-1}{2}} Q^{\frac{1}{2}}\|_{\mathcal{L}_2(H)} < \infty, \text{ for some } \beta \in (0, 1]. \quad (57)$$

Note that the semigroup proprieties in Proposition 2.2 are satisfied for the discrete operator A_h . For our convergence proof, we add the following propriety to the discrete operator A_h .

Proposition 4.1. Under Assumption 4.1 and Assumption 2.1, the following proprieties are satisfied for discrete operators

$$\|(-A_h)^{\frac{\beta-1}{2}} P_h Q^{\frac{1}{2}}\|_{\mathcal{L}_2(H)} < C + C_\beta h^{1-\beta}, \text{ for some } \beta \in (0, 1], \quad (58)$$

$$\|(-A_h)^{-\beta} P_h F''(Z)(X_1, X_2)\| \leq C(1 + h^{2\beta})\|X_1\|\|X_2\|, \quad Z, X_1, X_2 \in V_h; \quad \beta \in [0, 1], \quad (59)$$

$$\|(-A_h)^{-\frac{\gamma}{2}} P_h F'(Z)(X)\| \leq C(h^\gamma + (1 + \|Z\|_1)\|X\|_{-1}), \quad Z, X \in V_h; \quad \gamma \in [1, 2], \quad (60)$$

²This assumption is weak compared to Assumption 2.2 and Assumption 2.3

Proof. According to ([8], Theorem 5.2) for all $\alpha > 0$, the discrete operator A_h and the continuous operator A satisfy

$$\|A_h^{-\alpha}P_h - A^{-\alpha}\|_{L(H)} \leq C_\alpha h^{2\alpha}, \quad (61)$$

where C_α is a constant dependent on α . For $\alpha = -\frac{\beta-1}{2}$, with $\beta \in (0, 1]$, we have

$$\|(A_h^{-\alpha}P_h - A^{-\alpha})Q^{\frac{1}{2}}\|_{\mathcal{L}_2(H)} = \|A_h^{-\alpha}P_hQ^{\frac{1}{2}} - A^{-\alpha}Q^{\frac{1}{2}}\|_{\mathcal{L}_2(H)}$$

Now let $\{e_i\}_{i=1}^\infty$ be any orthonormal basis of H , since $\|A^{-\alpha}Q^{\frac{1}{2}}\|_{\mathcal{L}_2(H)} < \infty$, according to Assumption 4.1 (relation (57)), we have using (61)

$$\begin{aligned} \|A_h^{-\alpha}P_hQ^{\frac{1}{2}} - A^{-\alpha}Q^{\frac{1}{2}}\|_{\mathcal{L}_2(H)} &= \sum_{i=1}^\infty \|(A_h^{-\alpha}P_h - A^{-\alpha})Q^{\frac{1}{2}}e_i\| \\ &\leq \|A_h^{-\alpha}P_h - A^{-\alpha}\|_{L(H)} \|Q^{\frac{1}{2}}\|_{\mathcal{L}_2(H)} \\ &\leq C_\alpha h^{2\alpha}. \end{aligned}$$

Now we can write, using (61) and (57), that

$$\begin{aligned} \|A_h^{-\alpha}P_hQ^{\frac{1}{2}}\|_{\mathcal{L}_2(H)} &= \|A_h^{-\alpha}P_hQ^{\frac{1}{2}} - A^{-\alpha}Q^{\frac{1}{2}} + A^{-\alpha}Q^{\frac{1}{2}}\|_{\mathcal{L}_2(H)} \\ &\leq \|A_h^{-\alpha}P_hQ^{\frac{1}{2}} - A^{-\alpha}Q^{\frac{1}{2}}\|_{\mathcal{L}_2(H)} + \|A^{-\alpha}Q^{\frac{1}{2}}\|_{\mathcal{L}_2(H)} \\ &\leq C + C_\alpha h^{2\alpha} \\ &= C + C_\alpha h^{1-\beta}. \end{aligned}$$

This proves (58). Now to prove (59), we proceed as above and the fact that (55) is satisfied. Indeed for $\beta \in [0, 1)$, using (54), we have

$$\|(A_h^{-\beta}P_h - A^{-\beta})F''(Z)(X_1, X_2)\| \leq \|A_h^{-\beta}P_h - A^{-\beta}\|_{L(H)} \|F''(Z)(X_1, X_2)\| \leq CL h^{2\beta} \|X_1\| \|X_2\|.$$

Therefore,

$$\begin{aligned} \|A_h^{-\beta}P_hF''(Z)(X_1, X_2)\| &\leq \|(A_h^{-\beta}P_h - A^{-\beta})F''(Z)(X_1, X_2)\| + \|A^{-\beta}F''(Z)(X_1, X_2)\| \\ &\leq C(1 + h^{2\beta})\|X_1\| \|X_2\|. \end{aligned}$$

The proof of (60) is done just like for (59) using (56) and (53). ■

The estimates in Proposition 4.1 are very tight and can influence the order of convergence in space and time when β in (57) is small. Indeed using the fact that

$$h < C(\text{meas}(\Omega))^{\frac{1}{d}}, \quad (62)$$

where $\text{meas}(\Omega)$ is the either the length, the area or the volume of the domain Ω and d is the dimension, we have the following corollary.

Corollary 4.1. *Under Assumption 4.1 the following discrete proprieties are satisfied*

$$\|(-A_h)^{\frac{\beta-1}{2}}P_hQ^{\frac{1}{2}}\|_{\mathcal{L}_2(H)} < C, \quad \beta \in (0, 1]. \quad (63)$$

$$\|(-A_h)^{-\beta}P_hF''(Z)(X_1, X_2)\| \leq C\|X_1\| \|X_2\|, \quad Z, X_1, X_2 \in V_h; \quad \beta \in [0, 1), \quad (64)$$

$$\|(-A_h)^{-\frac{\gamma}{2}}P_hF'(Z)(X)\| \leq C(1 + \|Z\|_1)\|X\|_{-1}, \quad Z, X \in V_h; \quad \gamma \in [1, 2), \quad (65)$$

where the positive constant C is independent of h .

The following lemma will be helpful for the proofs of convergence.

Lemma 4.1. *Assume that all the conditions above are fulfilled and that $\Phi \in \mathcal{C}_2^b(H; \mathbb{R})$. For $\gamma \in [0, 1]$, $\gamma_1, \gamma_2 \in [0, 1)$ satisfying $\gamma_1 + \gamma_2 < 1$, there exists constants c_γ and c_{γ_1, γ_2} such that*

$$\|(-A_h)^\gamma D\mu^h(t, \psi)\| \leq c_\gamma t^{-\gamma}, \quad (66)$$

$$\|(-A_h)^{\gamma_2} D^2\mu^h(t, \psi)(-A_h)^{\gamma_1}\|_{L(H)} \leq c_{\gamma_1, \gamma_2}(t^{-(\gamma_1 + \gamma_2)} + 1), \quad (67)$$

where $\mu^h(t, \psi)$ is defined by (39), and $\psi \in V_h$.

Proof. The proof is the same as in [32]. Note that although the linear operator in [32] is assumed to be self adjoint, the proof don't make use of that. ■

The following results can be proven exactly along the line of [32, Lemma 3.4 and Lemma 3.5]).

Lemma 4.2. *Suppose Assumption 2.2 is fulfilled and $X_0 \in \mathcal{D}((-A)^{1/2})$, then it holds for $\gamma \in [0, \frac{\beta}{2})$ and arbitrary small ϵ that*

$$\sup_{0 \leq m \leq M} \|(-A_h)^\gamma X_m^h\|_{L_2(\mathbb{D}, H)} \leq C \text{ and } \|\tilde{X}^h(t) - X_m^h\|_{L_2(\mathbb{D}, H)} \leq C\Delta t^{\frac{\beta - \epsilon}{2}}, \quad (68)$$

where X_m^h is defined by (37) and \tilde{X}^h is given in (44). Furthermore

$$\|(-A_h)^{\frac{1}{2}} X_m^h\|_{L_2(\mathbb{D}, H)} \leq C(1 + \Delta t^{\frac{\beta - 1 - \epsilon}{2}}), \quad (69)$$

for $\beta \in (0, 1]$.

Proof. Before the proof of the lemma, we can prove exactly as in [32, (3.3)] that

$$\|X_k^h\|_{L_2(\mathbb{D}, H)} < \infty, \quad 0 \leq k \leq M, \quad \sup_{0 \leq k \leq M} \|P_h F(X_k^h)\|_{L_2(\mathbb{D}, H)} < \infty.$$

We concentrate on proving (69), the proof of the other assertion can be done just as in [32]. Recall that

$$X_m^h = e^{\Delta t A_h} X_{m-1}^h + A_h^{-1} (e^{\Delta t A_h} - I) P_h F(X_{m-1}^h) + e^{\Delta t A_h} B_h (W_{t_m} - W_{t_{m-1}}), \quad (70)$$

$$B_h = P_h Q^{1/2} P_h. \quad (71)$$

This can also be written as

$$X_m^h = S_h(\Delta t) X_{m-1}^h + \int_0^{\Delta t} S_h(\Delta t - s) P_h F(X_{m-1}^h) ds + S_h(\Delta t) B_h (W_{t_m} - W_{t_{m-1}}). \quad (72)$$

Iterating gives

$$X_m^h = S_h(t_m) X_0^h + \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} S_h(t_m - s) P_h F(X_k^h) ds + \sum_{k=0}^{m-1} S_h(t_m - t_k) B_h \Delta W_k. \quad (73)$$

Now

$$\begin{aligned} \|(-A_h)^{\frac{1}{2}} X_m^h\|_{L_2(\mathbb{D}, H)} &\leq \|(-A_h)^{\frac{1}{2}} S_h(t_m) X_0^h\| + \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} \|(-A_h)^{\frac{1}{2}} S_h(t_m - s) P_h F(X_k^h)\|_{L_2(\mathbb{D}, H)} ds \\ &\quad + \left(\Delta t \sum_{k=0}^{m-1} \|(-A_h)^{\frac{1}{2}} S_h(t_m - t_k) B_h\|_{\mathcal{L}_2(H)}^2 \right)^{\frac{1}{2}} \\ &= I_0 + I_1 + I_2. \end{aligned}$$

We can prove exactly as in ([32], Lemma 3.5) that

$$I_2 \leq C\Delta t^{\frac{\beta-1-\epsilon}{2}} \text{ and } I_0 \leq \|X_0\|_1 < \infty. \quad (74)$$

Now it remain to prove that I_1 is bounded above. We have

$$\begin{aligned} I_1 &\leq \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} \|(-A_h)^{\frac{1}{2}} S_h(t_m - s) P_h F(X_k^h) ds\|_{L_2(\mathbb{D}, H)} \\ &\leq C \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} (t_m - s)^{-\frac{1}{2}} (1 + \|X_k^h\|_{L_2(\mathbb{D}, H)}) ds \\ &\leq C \left(\sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} (t_m - s)^{-\frac{1}{2}} ds \right) \left(\sup_{0 \leq k \leq m-1} (1 + \|X_k^h\|_{L_2(\mathbb{D}, H)}) \right) \\ &= C \left(\int_0^{t_m} (t_m - s)^{-\frac{1}{2}} ds \right) \left(\sup_{0 \leq k \leq m-1} (1 + \|X_k^h\|_{L_2(\mathbb{D}, H)}) \right) \\ &\leq CT^{1/2} \left(\sup_{0 \leq k \leq m-1} (1 + \|X_k^h\|_{L_2(\mathbb{D}, H)}) \right) < \infty. \end{aligned}$$

Combining with (74) establishes the lemma. ■

Now we can prove the convergence result for the semi discrete problem with non-self-adjoint operator A as the time step goes to zero.

Theorem 4.1. *Assume that Assumption 2.2 is fulfilled $X_0 \in \mathcal{D}((-A)^{1/2})$ and that $\Phi \in \mathcal{C}_2^b(H; \mathbb{R})$. Then for arbitrary small $\epsilon > 0$, if the adjoint of the discrete operator A_h^* satisfies the analogue inequality as A_h in Corollary 4.1, we have*

$$|\mathbb{E}[\Phi(X_M^h)] - \mathbb{E}[\Phi(X^h(T))]| \leq C\Delta t^{\beta-\epsilon}, \quad (75)$$

where the constant C depends on $\beta, \epsilon, \dots, T, L$ and the initial data, but is independent of h and M .

Proof. We derived in (43) the following error representation

$$\mathbb{E}[\Phi(X_M^h)] - \mathbb{E}[\Phi(X^h(T))] = \sum_{m=0}^{M-1} (b_m^1 + b_m^2) \quad (76)$$

where the decompositions b_m^1 and b_m^2 are as follows

$$b_m^1 = \int_{t_m}^{t_{m+1}} \mathbb{E} \left[\langle D\mu^h(T-s, \tilde{X}^h(s)), P_h F(X_m^h) - P_h F(\tilde{X}^h(s)) \rangle_H \right] ds \quad (77)$$

and

$$\begin{aligned} b_m^2 &= \frac{1}{2} \mathbb{E} \left\{ \int_{t_m}^{t_{m+1}} \text{Tr} \left[D^2 \mu^h(T-s, \tilde{X}^h(s)) \left(\{S_h(s-t_m) B_h\} \{ (S_h(s-t_m) - I) B_h \}^* \right) \right] ds \right\} \\ &\quad + \frac{1}{2} \mathbb{E} \left\{ \int_{t_m}^{t_{m+1}} \text{Tr} \left[D^2 \mu^h(T-s, \tilde{X}^h(s)) (S_h(s-t_m) - I) B_h B_h^* \right] ds \right\} \\ &= b_m^{2,1} + b_m^{2,2}. \end{aligned} \quad (78)$$

Note that our term b_m^1 is more simple than the one in [32]. Applying the Taylor's formula to the drift term F we get

$$F(\tilde{X}^h(s)) - F(X_m^h) = F'(X_m^h)(\tilde{X}^h(s) - X_m^h) + \int_0^1 F''(\chi(r))(\tilde{X}^h(s) - X_m^h, \tilde{X}^h(s) - X_m^h)(1-r)dr,$$

where $\chi(r) = X_m^h + r(\tilde{X}^h(s) - X_m^h)$, which allows to have

$$\begin{aligned} |b_m^1| &\leq \left| \int_{t_m}^{t_{m+1}} \mathbb{E} \left[\left\langle D\mu^h(T-s, \tilde{X}^h(s)), P_h F'(X_m^h)(\tilde{X}^h(s) - X_m^h) \right\rangle_H \right] ds \right| \\ &\quad + \left| \int_{t_m}^{t_{m+1}} \mathbb{E} \left[\left\langle D\mu^h(T-s, \tilde{X}^h(s)), \int_0^1 P_h F''(\chi(r))(\tilde{X}^h(s) - X_m^h, \tilde{X}^h(s) - X_m^h)(1-r)dr \right\rangle_H \right] ds \right| \\ &= J_m^1 + J_m^2. \end{aligned}$$

To estimate J_m^2 , we use Corollary 4.1, Lemma 4.1, Lemma 4.2 and the Holder inequality

$$\begin{aligned} J_m^2 &\leq \int_{t_m}^{t_{m+1}} \mathbb{E} \left[\left\| \left\langle (-A_h)^{-\delta} (-A_h)^\delta D\mu^h(T-s, \tilde{X}^h(s)), \int_0^1 P_h F''(\chi(r))(\tilde{X}^h(s) - X_m^h, \tilde{X}^h(s) - X_m^h)(1-r)dr \right\rangle_H \right\| \right] ds \\ &= \int_{t_m}^{t_{m+1}} \mathbb{E} \left[\left\| \left\langle (-A_h)^\delta D\mu^h(T-s, \tilde{X}^h(s)), \int_0^1 (-A_h^*)^{-\delta} P_h F''(\chi(r))(\tilde{X}^h(s) - X_m^h, \tilde{X}^h(s) - X_m^h)(1-r)dr \right\rangle_H \right\| \right] ds \\ &\leq \int_{t_m}^{t_{m+1}} \mathbb{E} \left[\left\| (-A_h)^\delta D\mu^h(T-s, \tilde{X}^h(s)) \right\| \times \left\| \int_0^1 (-A_h^*)^{-\delta} P_h F''(\chi(r))(\tilde{X}^h(s) - X_m^h, \tilde{X}^h(s) - X_m^h)(1-r)dr \right\| \right] ds \\ &\leq c_\delta \int_{t_m}^{t_{m+1}} \int_0^1 \mathbb{E} \left[\left\| (-A_h^*)^{-\delta} P_h F''(\chi(r))(\tilde{X}^h(s) - X_m^h, \tilde{X}^h(s) - X_m^h) \right\| \right] (T-s)^{-\delta} dr ds \\ &\leq c_\delta L \int_{t_m}^{t_{m+1}} \int_0^1 \mathbb{E} \left[\left\| \tilde{X}^h(s) - X_m^h \right\|^2 \right] (T-s)^{-\delta} dr ds \\ &\leq C \Delta t^{\beta-\epsilon} \int_{t_m}^{t_{m+1}} (T-s)^{-\delta} ds. \end{aligned} \tag{79}$$

We now turn to estimate J_m^1 . Recall that from (44),

$$\begin{aligned} \tilde{X}^h(s) - X_m^h &= (S_h(s - t_m) - I)X_m^h + (s - t_m)\varphi_1^h(s - t_m)P_h F(X_m^h) \\ &\quad + S_h(s - t_m)B_h(W(s) - W(t_m)). \end{aligned} \tag{80}$$

Since the expectation of the Brownian motion vanishes, we therefore have

$$\begin{aligned}
J_m^1 &\leq \left| \int_{t_m}^{t_{m+1}} \mathbb{E} \left[\left\langle D\mu^h(T-s, \tilde{X}^h(s)), P_h F'(X_m^h)(S_h(s-t_m) - I) X_m^h \right\rangle_H \right] ds \right| \\
&\quad + \left| \int_{t_m}^{t_{m+1}} \mathbb{E} \left[\left\langle D\mu^h(T-s, \tilde{X}^h(s)), P_h F'(X_m^h)(S_h^1(s-t_m) P_h F(X_m^h)(s-t_m)) \right\rangle_H \right] ds \right| \\
&= I + II.
\end{aligned} \tag{81}$$

Using Lemma 4.1, Corollary 4.1, Lemma 4.2 and Proposition 4.1 yields

$$\begin{aligned}
I &\leq c_{\frac{\delta}{2}} \int_{t_m}^{t_{m+1}} \mathbb{E} \left[\left\| (-A_h^*)^{-\frac{\delta}{2}} P_h F'(X_m^h)(S_h(s-t_m) - I) X_m^h \right\| \right] (T-s)^{-\frac{\delta}{2}} ds \\
&\leq C \int_{t_m}^{t_{m+1}} \left[(1 + \|X_m^h\|_1) \left\| (S_h(s-t_m) - I) X_m^h \right\|_1 \right] (T-s)^{-\frac{\delta}{2}} ds \\
&\leq \int_{t_m}^{t_{m+1}} \left[1 + \mathbb{E} \|(-A_h)^{1/2} X_m^h\| \right] \mathbb{E} \|(-A_h)^{-\frac{1+\beta-\epsilon}{2}} S_h(s-t_m) - I\| (-A_h)^{\frac{\beta-\epsilon}{2}} X_m^h \| (T-s)^{-\frac{\delta}{2}} ds \\
&\leq \int_{t_m}^{t_{m+1}} \left[1 + \Delta t^{\frac{\beta-1-\epsilon}{2}} \right] (s-t_m)^{\frac{1+\beta-\epsilon}{2}} \mathbb{E} \|(-A_h)^{\frac{\beta-\epsilon}{2}} X_m^h\| (T-s)^{-\frac{\delta}{2}} ds \\
&\leq C \Delta t^{\beta-\epsilon} \int_{t_m}^{t_{m+1}} (T-s)^{-\frac{\delta}{2}} ds.
\end{aligned}$$

Now we turn to approximate the term II . Note that we can rewrite (80) as

$$\begin{aligned}
\tilde{X}^h(s) - X_m^h &= (S_h(s-t_m) - I) X_m^h + \int_{t_m}^s S_h(t-s) P_h F(X_m^h) dt \\
&\quad + S_h(s-t_m) B_h(W(s) - W(t_m)).
\end{aligned}$$

Therefore

$$\begin{aligned}
II &= \left| \int_{t_m}^{t_{m+1}} \mathbb{E} \left[\left\langle D\mu^h(T-s, \tilde{X}^h(s)), \int_{t_m}^s P_h F'(X_m^h) S_h(t-s) P_h F(X_m^h) dt \right\rangle_H \right] ds \right| \\
&\leq \int_{t_m}^{t_{m+1}} \int_{t_m}^s \mathbb{E} \left[\|D\mu^h(T-s, \tilde{X}^h(s))\| \|P_h F'(X_m^h) S_h(t-s) P_h F(X_m^h)\| \right] dt ds \\
&\leq C \int_{t_m}^{t_{m+1}} \int_{t_m}^s \mathbb{E} \left[\|S_h(t-s) P_h F(X_m^h)\| \right] dt ds \\
&\leq C \mathbb{E} [\|P_h F(X_m^h)\|] \int_{t_m}^{t_{m+1}} \int_{t_m}^s dt ds \\
&\leq C \Delta t^2.
\end{aligned} \tag{82}$$

Combining (76), (82) and (82), we have

$$|b_m^1| \leq C \Delta t^{\beta-\epsilon} \int_{t_m}^{t_{m+1}} (T-s)^{-\frac{\delta}{2}} ds + C \Delta t^2 \tag{83}$$

$$\sum_{m=0}^{M-1} |b_m^1| \leq C \Delta t^{\beta-\epsilon} \tag{84}$$

The term b_m^2 can be approximated just as in [32] to be

$$\sum_{m=0}^{M-1} |b_m^2| \leq C\Delta t^{\beta-\epsilon}. \quad (85)$$

Finally we therefore have

$$|\mathbb{E}[\Phi(X_M^h)] - \mathbb{E}[\Phi(X^h(T))]| \leq C\Delta t^{\beta-\epsilon}. \quad (86)$$

■

5. Weak convergence for a SPDE with multiplicative noise

Now we consider the case of a stochastic partial differential equation with multiplicative noise, that is (1)

For convergence proof, we make the following assumption on the noise term.

Assumption 5.1. *We assume that there exists a constant $\alpha > 0$ such that*

$$\|(-A)^{\frac{\beta-1}{2}} B(X)\|_{\mathcal{L}_2(H)} \leq C(1 + \|X\|_{\beta-1}), \text{ for all } X \in H, \beta \in [0, 1]. \quad (87)$$

One can prove just as in the proof of Proposition 4.1 that the discrete operators A_h and $P_h B(X)$ satisfies

$$\|(-A_h)^{\frac{\beta-1}{2}} P_h B(X)\|_{\mathcal{L}_2(H)} \leq Ch^{1-\beta}(1 + \|X\|_{\beta-1}) \leq C'(1 + \|X\|_{\beta-1}), \text{ for all } X \in H. \quad (88)$$

For $\beta \in [0, 1]$, we also have using Proposition 2.1

$$\begin{aligned} \|(-A_h)^{\frac{\beta-1}{2}} B_h(X)\|_{\mathcal{L}_2(H)} &= \|(-A_h)^{\frac{\beta-1}{2}} P_h B(X) P_h\|_{\mathcal{L}_2(H)} \\ &\leq C \|(-A_h)^{\frac{\beta-1}{2}} P_h B(X)\|_{\mathcal{L}_2(H)} \\ &\leq C'(1 + \|X\|_{\beta-1}), \\ &\leq C'(1 + \|X\|), \text{ for all } X \in H. \end{aligned} \quad (89)$$

Thanks to Lemma 4.2 and (88) (or (89)) all the results presented above for additive noise also hold for multiplicative noise and we then have the following convergence result.

Theorem 5.1. *Assume that all the conditions of Theorem 4.1 are satisfied with the condition (58) replaced by (88). For $X_0 \in \mathcal{D}((-A)^{1/2})$, $\Phi \in \mathcal{C}_b^1(H; \mathbb{R})$ and for arbitrary small $\epsilon > 0$, if the adjoint of the discrete operator A_h^* satisfies the analogue inequality as A_h in Corollary 4.1, we have the following weak error convergence rate*

$$|\mathbb{E}[\Phi(X_M^h)] - \mathbb{E}[\Phi(X^h(T))]| \leq C\Delta t^{\beta-\epsilon}, \quad (90)$$

where the constant C depends on $\beta, \epsilon, \dots, T, L$ and the initial data, but is independent of h and M .

Proof. The proof follows the same lines as that of Theorem 4.1

$$\mathbb{E}[\Phi(X_M^h)] - \mathbb{E}[\Phi(X^h(T))] = \sum_{m=0}^{M-1} (b_m^1 + b_m^2). \quad (91)$$

The term b_m^1 is the same as in Theorem 4.1. Here b_m^2 is given by

$$\begin{aligned} b_m^2 &= \frac{1}{2} \mathbb{E} \left[\int_{t_m}^{t_{m+1}} \text{Tr} \left[D^2 \mu^h(T-s, \tilde{X}^h(s)) (S_h(s-t_m) B_h(X_m^h)) \right. \right. \\ &\quad \left. \left. \times ((S_h(s-t_m) - I) B_h(X_m^h))^* \right] ds \right] \\ &\quad + \frac{1}{2} \mathbb{E} \left[\int_{t_m}^{t_{m+1}} \text{Tr} \left[D^2 \mu^h(T-s, \tilde{X}^h(s)) (S_h(s-t_m) - I) (B_h(X_m^h)) (B_h(X_m^h))^* \right] ds \right] \\ &= b_m^{2,1} + b_m^{2,2}. \end{aligned} \quad (92)$$

Let us estimate $b_m^{2,1}$. Proposition 2.1 allows to have

$$\begin{aligned} b_m^{2,1} &\leq \frac{1}{2} \mathbb{E} \left[\int_{t_m}^{t_{m+1}} \left\| D^2 \mu^h(T-s, \tilde{X}^h(s)) [S_h(s-t_m) B_h(X_m^h)] \right. \right. \\ &\quad \left. \left. \times [(S_h(s-t_m) - I) B_h(X_m^h)]^* \right\|_{\mathcal{L}_1(H)} ds \right] \\ &= \frac{1}{2} \mathbb{E} \left[\int_{t_m}^{t_{m+1}} \left\| (-A_h)^{\frac{\beta+1}{2}-\epsilon} D^2 \mu^h(T-s, \tilde{X}^h(s)) (-A_h)^{\frac{1-\beta}{2}} (-A_h)^{\frac{\beta-1}{2}} [S_h(s-t_m) B_h(X_m^h)] \right. \right. \\ &\quad \left. \left. \times [(-A_h)^{-\frac{\beta+1}{2}+\epsilon} (S_h(s-t_m) - I) B_h(X_m^h)]^* \right\|_{\mathcal{L}_1(H)} ds \right] \end{aligned}$$

Using Lemma 4.1, Proposition 2.1, Proposition 2.2 and (89) yields the following estimations

$$\begin{aligned} b_m^{2,1} &\leq C \mathbb{E} \left[\int_{t_m}^{t_{m+1}} \left\| (-A_h)^{\frac{\beta-1}{2}} S_h(s-t_m) B_h(X_m^h) \left[(-A_h)^{\frac{\beta+1}{2}-\epsilon} (S_h(s-t_m) - I) B_h(X_m^h) \right]^* \right\|_{\mathcal{L}_1(H)} \right. \\ &\quad \left. \times [(T-s)^{-1+\epsilon} + 1] ds \right] \\ &\leq C \mathbb{E} \left[\int_{t_m}^{t_{m+1}} \left\| (-A_h)^{\frac{\beta-1}{2}} S_h(s-t_m) B_h(X_m^h) \right\|_{\mathcal{L}_2(H)} \right. \\ &\quad \left. \times \left\| (-A_h)^{-\frac{\beta+1}{2}+\epsilon} (S_h(s-t_m) - I) B_h(X_m^h) \right\|_{\mathcal{L}_2(H)} [(T-s)^{-1+\epsilon} + 1] ds \right] \\ &\leq C \mathbb{E} \left[\int_{t_m}^{t_{m+1}} \left\| (-A_h)^{\frac{\beta-1}{2}} B_h(X_m^h) \right\|_{\mathcal{L}_2(H)} \right. \\ &\quad \left. \times \left\| (-A_h)^{-\beta+\epsilon} (S_h(s-t_m) - I) (-A_h)^{\frac{\beta-1}{2}} B_h(X_m^h) \right\|_{\mathcal{L}_2(H)} [(T-s)^{-1+\epsilon} + 1] ds \right] \\ &\leq C \left[\int_{t_m}^{t_{m+1}} \sup_{0 \leq i \leq M} \mathbb{E}(1 + \|X_i^h\|)^2 \cdot \left\| (-A_h)^{-\beta+\epsilon} (S_h(s-t_m) - I) \right\|_{L(H)} [(T-s)^{-1+\epsilon} + 1] ds \right] \\ &\leq C \Delta t^{\beta-\epsilon} \int_{t_m}^{t_{m+1}} (T-s)^{-1+\epsilon} ds + C \Delta t^{\beta-\epsilon+1}. \end{aligned}$$

Similar as for $b_m^{2,2}$, we have

$$\begin{aligned}
b_m^{2,2} &= \frac{1}{2} \mathbb{E} \left[\int_{t_m}^{t_{m+1}} \text{Tr} \left[(-A_h)^{\frac{1-\beta}{2}} D^2 \mu^h(T-s, \tilde{X}^h(s)) (-A_h)^{\frac{\beta+1}{2}-\epsilon} \right. \right. \\
&\quad \left. \left. \times (-A_h)^{-\frac{\beta+1}{2}+\epsilon} (S_h(s-t_m) - I) (B_h(X_m^h)) \left((-A_h)^{\frac{\beta-1}{2}} B_h(X_m^h) \right)^* \right] ds \right] \\
&\leq C \mathbb{E} \left[\int_{t_m}^{t_{m+1}} \left\| (-A_h)^{-\beta+\epsilon} (S_h(s-t_m) - I) (-A_h)^{\frac{\beta-1}{2}} B_h(X_m^h) \left[(-A_h)^{\frac{\beta-1}{2}} B_h(X_m^h) \right]^* \right\|_{\mathcal{L}_1(H)} \right. \\
&\quad \left. \times [(T-s)^{-1+\epsilon} + 1] ds \right] \\
&\leq C \Delta t^{\beta-\epsilon} \int_{t_m}^{t_{m+1}} (T-s)^{-1+\epsilon} ds + C \Delta t^{\beta-\epsilon+1}.
\end{aligned}$$

So by summing up as in Theorem 4.1, the proof is ended. \blacksquare

6. Strong convergence and toward full weak convergence results

The goal here is to provide the space and time convergence proof of the exponential scheme. Before that we will provide strong convergence of the semi discrete solution.

Theorem 6.1. *Let X^h and \bar{X}_h be the solutions respectively of (35) and the following semi discrete problem*

$$\begin{aligned}
d\bar{X}_h &= (A_h \bar{X}_h + P_h F(\bar{X}_h)) dt + P_h B(\bar{X}_h) dW \\
\bar{X}_h(0) &= P_h X_0.
\end{aligned} \tag{93}$$

Let $\beta \in [0, 2)$, assume that Assumption 2.2 and Assumption 2.3 are satisfied. For $\beta \in [0, 1]$ assume that the relation (57) of Assumption 4.1 (when dealing with additive noise) and Assumption 5.1 (when dealing with multiplicative noise) are satisfied. For $\beta \in [1, 2)$ assume that $B(\mathcal{D}((-A)^{\frac{\beta-1}{2}})) \subset HS \left(Q^{1/2}(H), \mathcal{D}((-A)^{\frac{\beta-1}{2}}) \right)$ and $\|(-A)^{\frac{\beta-1}{2}} B(v)\|_{L_0^2} \leq c(1 + \|v\|_{\beta-1})$ for $v \in \mathcal{D}((-A)^{\frac{\beta-1}{2}})$. If $X_0 \in L_2(\mathbb{D}, \mathcal{D}((-A)^{\beta/2}))$, there exists a positive constant C independent of h such that the following estimations hold:

$$\|X(t) - \bar{X}_h(t)\|_{L_2(\mathbb{D}, H)} \leq Ch^\beta, \quad \beta \in [0, 1]. \tag{94}$$

Furthermore assume that the linear operator A is self adjoint, the following estimation hold

$$\|X(t) - X^h(t)\|_{L_2(\mathbb{D}, H)} \leq Ch^\beta, \quad \beta \in [0, 2). \tag{95}$$

Before prove Theorem 6.1 let us make some remarks and provide some preparatory results.

Remark 6.1. *Theorem 6.1 extends [19, Theorem 1.1] for non-self-adjoint operator A and also provide optimal convergence proof for $\beta \in [0, 1)$ which was not studied in [19].*

Remark 6.2. For additive noise, we can observe from [18] that the semi discrete problem (35) is equivalent to the following problem, find $X_h(t) \in V_h$ such that

$$\begin{aligned} dX_h &= (A_h X_h + P_h F(X_h))dt + dW_h \\ X_h(0) &= P_h X_0. \end{aligned} \quad (96)$$

where $W_h(t)$ is a $P_h Q P_h$ -Wiener process on V_h with the following representation

$$W_h(t) = \sum_{i=1}^{N_h} \sqrt{q_{h,i}} e_{h,i} \beta_i(t), \quad (97)$$

where $(q_{h,i}, e_{h,i})$ are the eigenvalues and eigenfunctions of the covariance operator $Q_h := P_h Q P_h$ and β_i are independent and identically distributed standard Brownian motions. More precisely $(q_{h,i}, e_{h,i})$ is the finite element solution of the eigenvalue problem $Qu = \gamma u$.

If the exact eigenvalues and eigenfunctions (q_i, e_i) of the covariance operator Q are known, replacing in (96) (or (35)) $W_h(t)$ by $W_h^{N_h}(t)$, defined by

$$W_h^{N_h}(t) = \sum_{i=1}^{N_h} \sqrt{q_i} e_i \beta_i(t), \quad (98)$$

will not necessarily change the optimal convergence order in our scheme. From [18], it is also proved that if the kernel of the covariance function Q is regular and the mesh family is quasi-uniform, it is enough to take $M < N_h$ noise terms in (98) (or (97)) without loss the optimal convergence order.

Of course for multiplicative noise $P_h W$ can also be expanded on the basis of V_h with N_h terms (see [35]).

As we are also dealing in Theorem 6.1 with non-self-adjoint operator in (94), let us provide some preparatory results before giving the proof of Theorem 6.1.

We introduce the Riesz representation operator $R_h : V \rightarrow V_h$ defined by

$$(-AR_h v, \chi) = (-Av, \chi) = a(v, \chi), \quad v \in V, \quad \forall \chi \in V_h.$$

Under the usual regularity assumptions on the triangulation and in view of V -ellipticity (21), it is well known (see [8]) that the following error bounds holds

$$\|R_h v - v\| + h \|R_h v - v\|_{H^1(\Omega)} \leq Ch^r \|v\|_{H^r(\Omega)}, \quad v \in V \cap H^r(\Omega), \quad r \in \{1, 2\}. \quad (99)$$

By interpolation, we have

$$\|R_h v - v\| + h \|R_h v - v\|_{H^1(\Omega)} \leq Ch^r \|v\|_r, \quad v \in \mathcal{D}(A^{r/2}), \quad 1 \leq r \leq 2. \quad (100)$$

Let us consider the following deterministic problem, which consists of finding $u \in V$ such that such that

$$u' = Au \quad \text{given} \quad u(0) = v, \quad t \in (0, T]. \quad (101)$$

The corresponding semi-discretization in space is : Find $u_h \in V_h$ such that

$$u'_h = A_h u_h$$

where $u_h^0 = P_h v$. Define the operator

$$T_h(t) := S(t) - S_h(t)P_h = e^{tA} - e^{tA_h}P_h \quad (102)$$

so that $u(t) - u_h(t) = T_h(t)v$.

The following lemma will be important in our proof.

Lemma 6.1. *The following estimates hold on the semi-discrete approximation of (101). There exists a constant $C > 0$ such that*

- (i) For $v \in \mathcal{D}((-A)^{\gamma/2})$

$$\|u(t) - u_h(t)\| = \|T_h(t)v\| \leq Ch^r t^{-(r-\gamma)/2} \|v\|_{\gamma}, \quad 1 \leq r \leq 2, \quad 0 \leq \gamma \leq r. \quad (103)$$

- (ii) For $v \in \mathcal{D}((-A)^{(\gamma-1)/2})$

$$\left(\int_0^t \|T_h(s)v\|^2 ds \right)^{\frac{1}{2}} \leq Ch^{\gamma} \|v\|_{\gamma-1}, \quad 0 \leq \gamma \leq 2. \quad (104)$$

Proof. The proof of (i) can be found in [22] using (100). For self adjoint operator, the proof of (ii) is done as in [34, Lemme 4.1] if $\gamma \in [0, 1]$ and in [19, Lemma 4.2] if $\gamma \in [1, 2)$ and the parameter r used in [19, Lemma 4.2] is $r = \gamma - 1$. Both proofs only uses general concepts and not spectral decomposition of the linear operator A , so can easily be generalized. Note that the non-self adjoint case should make use of [31, 4.17] or [31, Lemma 4.3] instead of [31, 2.29] used in [19, Lemma 4.2].

■

Let us now provide the proof of Theorem 6.1.

Proof. The proof of the estimation (95) for additive noise can be found in [18] and can be updated to multiplicative noise following [35].

For $\beta \in [1, 2)$, the proof for the estimation (94) when A is self adjoint operator can be found in [19] as the Assumption 2.2 implies [19, Assumption 2.1] by taking $r = \beta - 1$ in [19, Theorem 1.1]. Let us give more general proof by closely follow [18, 19]. The corresponding mild solution of (93) is given by

$$\bar{X}_h(t) = S_h(t)P_h X_0 + \int_0^t S_h(t-s)P_h F(\bar{X}_h(s))ds + \int_0^t S_h(t-s)P_h B(\bar{X}_h(s))dW(s). \quad (105)$$

Indeed, we have

$$\begin{aligned} & \|X(t) - \bar{X}_h(t)\|_{L_2(\mathbb{D}, H)} \\ & \leq \|T_h(t)P_h X_0 + \int_0^t S(t-s)F(X(s))ds - \int_0^t S_h(t-s)P_h F(\bar{X}_h(s))ds\|_{L_2(\mathbb{D}, H)} \\ & \quad + \left\| \int_0^t S(t-s)B(X(s))dW(s) - \int_0^t S_h(t-s)P_h B(\bar{X}_h(s))dW(s) \right\|_{L_2(\mathbb{D}, H)} \\ & = I_1 + I_2. \end{aligned} \quad (106)$$

The estimation of I_1 is the same as in [18] and we have

$$I_1 \leq C \int_0^t \|X(s) - \bar{X}_h(s)\|_{L_2(\mathbb{D}, H)} ds + Ch^\beta. \quad (107)$$

For the estimation of I_2 , we follow closely [19]. Indeed we have

$$I_2 \leq C \left(\mathbb{E} \left[\int_0^t \|S(t-s)B(X(s)) - S_h(t-s)P_h B(\bar{X}_h(s))\|_{\mathcal{L}_2(H)}^2 ds \right] \right)^{1/2} \quad (108)$$

$$\leq C \left\| \left(\int_0^t \|S_h(t-s)P_h(B(\bar{X}_h(s)) - B(X(s)))\|_{\mathcal{L}_2(H)}^2 ds \right)^{1/2} \right\|_{L_2(\mathbb{D}, \mathbb{R})} \quad (109)$$

$$+ C \left\| \left(\int_0^t \|T_h(t-s)(B(X(s)) - B(X(t)))\|_{\mathcal{L}_2(H)}^2 ds \right)^{1/2} \right\|_{L_2(\mathbb{D}, \mathbb{R})} \quad (110)$$

$$+ C \left\| \left(\int_0^t \|T_h(t-s)B(X(t))\|_{\mathcal{L}_2(H)}^2 ds \right)^{1/2} \right\|_{L_2(\mathbb{D}, \mathbb{R})} \quad (111)$$

$$= I_2^1 + I_2^2 + I_2^3. \quad (112)$$

The stability propriety of the semi group Proposition 2.2 and the Lipschitz condition in Assumption 2.2 allow to have

$$I_2^1 \leq C \left(\int_0^t \|X(s) - \bar{X}_h(s)\|_{L_2(\mathbb{D}, H)}^2 ds \right)^{\frac{1}{2}}. \quad (113)$$

Following closely [19], but with (103) in Lemma 6.1 with $r = \beta$, $\gamma = 0$, allow to have

$$I_2^2 \leq Ch^\beta. \quad (114)$$

For $\beta \in [0, 2)$, as in [19], by using (ii) in Lemma 6.1 gives

$$I_2^3 \leq Ch^\beta. \quad (115)$$

Coming back to I_2 , we have

$$\left\| \int_0^t S(t-s)B(X(s))dW(s) - \int_0^t S_h(t-s)P_h B(\bar{X}_h(s))dW(s) \right\|_{L_2(\mathbb{D}, H)} \quad (116)$$

$$\leq Ch^{2-\epsilon} + C \left(\int_0^t \|X(s) - \bar{X}_h(s)\|_{L_2(\mathbb{D}, H)}^2 ds \right)^{\frac{1}{2}} \quad (117)$$

$$\leq Ch^\beta + C \left(\int_0^t \|X(s) - \bar{X}_h(s)\|_{L_2(\mathbb{D}, H)}^2 ds \right)^{\frac{1}{2}}. \quad (118)$$

Combining I_1 and I_2 gives

$$\|X(t) - \bar{X}_h(t)\|_{L_2(\mathbb{D}, H)}^2 \leq Ch^{2\beta} + C \int_0^t \|X(s) - \bar{X}_h(s)\|_{L_2(\mathbb{D}, H)}^2 ds. \quad (119)$$

Gronwall's lemma is therefore applied to end the proof. ■

The following theorem provide the full weak convergence when the solution is regular enough.

Theorem 6.2. *Let X and X_M^h be respectively the solution of (1) and the numerical solution from (38) at the final time T . Let $\beta \in [1, 2)$, assume that Assumption 2.2 (for multiplicative noise, all conditions except (57)) and Assumption 2.3 (for multiplicative noise) are satisfied. For $\beta = 1$ (trace class noise) assume (57) of Assumption 4.1 (when dealing with additive noise) and Assumption 5.1 (when dealing with multiplicative noise) are satisfied. For $\beta \in (1, 2)$ assume that (57) of Assumption 4.1 is also satisfied for additive noise, but $B(\mathcal{D}((-A)^{\frac{\beta-1}{2}})) \subset HS\left(Q^{1/2}(H), \mathcal{D}((-A)^{\frac{\beta-1}{2}/2})\right)$ and, $\|(-A)^{\frac{\beta-1}{2}}B(v)\|_{L_0^2} \leq c(1 + \|v\|_{\beta-1})$ for $v \in \mathcal{D}((-A)^{\frac{\beta-1}{2}})$ for multiplicative noise. Furthermore assume that $X_0 \in \mathcal{D}((-A)^{\beta/2})$ and the linear operator is selfadjoint. For $\Phi \in \mathcal{C}_2^b(H; \mathbb{R})$ and arbitrary small $\epsilon > 0$, the following estimation hold*

$$|\mathbb{E}[\Phi(X_M^h)] - \mathbb{E}[\Phi(X(T))]| \leq C(\Delta t^{1-\epsilon} + h^\beta), \quad (120)$$

where the constant C depends on $\alpha, \epsilon, \dots, T, L$ and the initial data, but is independent of h and M .

Proof. Indeed we have the following decomposition

$$|\mathbb{E}[\Phi(X_M^h)] - \mathbb{E}[\Phi(X(T))]| \leq |\mathbb{E}[\Phi(X_M^h)] - \mathbb{E}[\Phi(X^h(T))]| + |\mathbb{E}[\Phi(X^h(T)) - \mathbb{E}[\Phi(X(T))]|. \quad (121)$$

Note that if $X_0 \in \mathcal{D}((-A)^{\beta/2})$, $1 \leq \beta < 2$, then $X_0 \in \mathcal{D}((-A)^{1/2})$ and all the lemma used in Theorem 4.1 and Theorem 5.1 are still valid. From Theorem 4.1 and Theorem 5.1 with $\beta = 1$ (as the noise is assumed to be trace class), we have

$$|\mathbb{E}[\Phi(X_M^h)] - \mathbb{E}[\Phi(X^h(T))]| \leq C\Delta t^{1-\epsilon}. \quad (122)$$

Note that this temporal order is the double of the strong convergence obtained in [22]. Remember that we are using low order finite element method for space discretization, where the optimal convergence is 2 (for deterministic case), therefore for $X_0 \in \mathcal{D}((-A)^{\beta/2})$, $1 < \beta < 2$, according to Theorem 6.1 and Remark 6.2, we cannot expect the order of weak convergence to double the strong order in space. The weak convergence order in space (of course, not necessarily optimal) can be the same as the strong convergence order when the solution is regular enough. Using the fact that $\Phi \in \mathcal{C}_2^b(H; \mathbb{R})$, so is Lipschitz, we therefore have

$$\begin{aligned} |\mathbb{E}[\Phi(X^h(T)) - \mathbb{E}[\Phi(X(T))]| &\leq |\mathbb{E}[\Phi(X^h(T)) - \Phi(X(T))]| \\ &\leq \mathbb{E}\|X^h(T) - X(T)\| \\ &\leq C\|X^h(T) - X(T)\|_{L_2(\mathbb{D}, H)} \\ &\leq Ch^\beta. \end{aligned}$$

■

The following theorem using recent result in the literature shows that the space order of convergence in Theorem 6.2 is far to be optimal for $\beta = 1$.

Theorem 6.3. *Assume that $A = \Delta$ with Dirichlet boundary condition ($V = H_0^1(\Omega)$), assume that the noise is additive, Assumption 4.1 is satisfied with $\|(-A)^{\frac{\beta-1}{2}}Q^{1/2}\|_{\mathcal{L}_2(H)} < \infty$ for some $\beta \in [1/2, 1]$. Furthermore assume that $X_0 \in \mathcal{D}((-A)^\beta)$. For $\Phi \in \mathcal{C}_2^b(H; \mathbb{R})$ and arbitrary small $\epsilon > 0$, the following estimation hold*

$$|\mathbb{E}[\Phi(X_M^h)] - \mathbb{E}[\Phi(X(T))]| \leq C(\Delta t^{\beta-\epsilon} + h^{2\beta-\epsilon}), \quad (123)$$

where the constant C depends on $\alpha, \epsilon, \dots, T, L$ and the initial data, but is independent of h and M .

Let us prove Theorem 6.3.

Proof. The proof follows the one for Theorem 6.2 and we have

$$|\mathbb{E}[\Phi(X_M^h)] - \mathbb{E}[\Phi(X(T))]| \leq |\mathbb{E}[\Phi(X_M^h)] - \mathbb{E}[\Phi(X^h(T))]| + |\mathbb{E}[\Phi(X^h(T)) - \mathbb{E}[\Phi(X(T))]|. \quad (124)$$

If $X_0 \in \mathcal{D}((-A)^\beta)$, $1/2 \leq \beta \leq 1$, then $X_0 \in \mathcal{D}((-A)^{1/2})$ and all the lemma used in Theorem 4.1 and Theorem 5.1 are still valid. From Theorem 4.1, if $\|(-A)^{\frac{\beta-1}{2}}Q^{1/2}\|_{\mathcal{L}_2(H)} < \infty$ for some $\beta \in [1/2, 1]$ we have

$$|\mathbb{E}[\Phi(X_M^h)] - \mathbb{E}[\Phi(X^h(T))]| \leq C\Delta t^{\beta-\epsilon}. \quad (125)$$

Note that Assumption 4.1 implies that $F \in \mathcal{C}_2^b(H; H)$. For $\|(-A)^{\frac{\beta-1}{2}}Q^{1/2}\|_{\mathcal{L}_2(H)} < \infty$ for some $\beta \in [1/2, 1]$, [2, Assumption A, Theorem 1.1] gives

$$|\mathbb{E}[\Phi(X^h(T)) - \mathbb{E}[\Phi(X(T))]| \leq h^{2\beta-\epsilon}. \quad (126)$$

The proof of (126) in [2] uses some elements of Malliavin calculus. ■

The following remark generalizes the Theorem 6.3 for general selfadjoint operator with not necessarily Dirichet boundary condition.

Remark 6.3. For additive noise and under the same condition as in Theorem 6.3, if A is selfadjoint, and $\|(-A)^{\frac{\beta-1}{2}}Q^{1/2}\|_{\mathcal{L}_2(H)} < \infty$ for some $\beta \in [k, 1]$, $k \geq 1/2$ and $X_0 \in \mathcal{D}((-A)^\beta)$. For $\Phi \in \mathcal{C}_2^b(H; \mathbb{R})$ and arbitrary small $\epsilon > 0$, the following estimation hold

$$|\mathbb{E}[\Phi(X_M^h)] - \mathbb{E}[\Phi(X(T))]| \leq C(\Delta t^{\beta-\epsilon} + h^{2\beta-\epsilon}). \quad (127)$$

The proof is the same as in [2, Assumption A, Theorem 1.1] where the Laplace operator is used just for simplicity. The difference comes from the set of the eigenvalues of the self adjoint operator A . Once the range of β such that $\|(-A)^{\frac{\beta-1}{2}}Q^{1/2}\|_{\mathcal{L}_2(H)} < \infty$ is found, the proof of [2, Assumption A, Theorem 1.1] is applied line by line.

The following numerical simulations will confirm numerically estimation (127) of Remark 6.3.

7. Numerical Simulations

We consider the reaction diffusion equation

$$dX = (D\Delta X - 0.5X)dt + dW \quad \text{given} \quad X(0) = X_0, \quad (128)$$

on the time interval $[0, T]$ and homogeneous Neumann boundary conditions on the domain $\Omega = [0, L_1] \times [0, L_2]$. The noise is represented by (2). The eigenfunctions $\{e_{i,j}\}_{i,j \geq 0} = \{e_i^{(1)} \otimes e_j^{(2)}\}_{i,j \geq 0}$ of the operator $-\Delta$ here are given by

$$e_0^{(l)}(x) = \sqrt{\frac{1}{L_l}}, \quad e_i^{(l)}(x) = \sqrt{\frac{2}{L_l}} \cos(\lambda_i^{(l)} x), \quad \lambda_0^{(l)} = 0, \quad \lambda_i^{(l)} = \frac{i\pi}{L_l}$$

where $l \in \{1, 2\}$, $x \in \Omega$ and $i \in \mathbb{N}$ with the corresponding eigenvalues $\{\lambda_{i,j}\}_{i,j \geq 0}$ given by $\lambda_{i,j} = (\lambda_i^{(1)})^2 + (\lambda_j^{(2)})^2$. We take $L_1 = L_2 = 1$. Notice that $A = D\Delta$ does not satisfy Assumption 2.1 as 0 is an eigenvalue. To eliminate the eigenvalue 0 we use the perturbed operator $A = D\Delta + \epsilon \mathbf{I}$, $\epsilon > 0$. The exact solution of (128) is known. Indeed the decomposition of (128) in each eigenvector node yields the following Ornstein-Uhlenbeck process

$$dX_i = -(D\lambda_i + 0.5)X_i dt + \sqrt{q_i} d\beta_i(t) \quad i \in \mathbb{N}^2. \quad (129)$$

This is a Gaussian process with the mild solution

$$X_i(t) = e^{-k_i t} X_i(0) + \sqrt{q_i} \int_0^t e^{k_i(s-t)} d\beta_i(s), \quad k_i = D\lambda_i + 0.5, \quad (130)$$

which is therefore an Ornstein-Uhlenbeck process. Applying the Itô isometry yields the following exact variance of $X_i(t)$

$$\text{Var}(X_i(t)) = \frac{q_i}{2k_i} (1 - e^{-2k_i t}). \quad (131)$$

During the simulations, we compute the exact solution recurrently as

$$\begin{aligned} X_i^{m+1} &= e^{-k_i \Delta t} X_i^m + \sqrt{q_i} \int_{t_m}^{t_{m+1}} e^{k_i(s-t)} d\beta_i(s) \\ &= e^{-k_i \Delta t} X_i^m + \left(\frac{q_i}{2k_i} (1 - e^{-2k_i \Delta t}) \right)^{1/2} R_{i,m}, \end{aligned} \quad (132)$$

where $R_{i,m}$ are independent, standard normally distributed random variables with mean 0 and variance 1. The expression in (132) allows to use the same set of random numbers for both the exact and the numerical solutions.

Our function $F(u) = -0.5u$ is linear and obviously satisfies the Lipschitz condition in Assumption 2.2. We assume that the covariance operator Q and A have the same eigenfunctions and we take the eigenvalues of the covariance operator to be

$$q_{i,j} = (i^2 + j^2)^{-(\beta+\delta)}, \beta > 0 \quad (133)$$

in the representation (2) for some small $\delta > 0$. Indeed for $\beta \in [0, 1]$ we have

$$\|(-A)^{\frac{\beta-1}{2}} Q^{1/2}\|_{\mathcal{L}_2(H)} < \infty \Leftrightarrow \sum_{(i,j) \in \mathbb{N}^2} \lambda_{i,j}^{\beta-1} q_{i,j} < \pi^2 \sum_{(i,j) \in \mathbb{N}^2} (i^2 + j^2)^{-(1+\delta)} < \infty.$$

We will consider the following two test functions

$$\Phi_1 : f \rightarrow \int_{\Omega} f(x) dx, \quad \Phi_2 : f \rightarrow \|f\|^2. \quad (134)$$

which obviously belong to $\mathcal{C}_b^2(H, \mathbb{R})$. By setting $B = D\Delta - 0.5I$, using the fact the the Ito's

integral vanishes, for $X_0 \in H$ we have

$$\begin{aligned}
\mathbb{E}\Phi_1(X(t)) &= \int_{\Omega} e^{tB} X_0(x) dx + \mathbb{E} \left[\int_{\Omega} \left(\int_0^t e^{(t-s)B} dW(s, x) \right) dx \right] \\
&= \int_{\Omega} e^{tB} X_0(x) dx + \int_{\Omega} \mathbb{E} \left[\int_0^t e^{(t-s)B} dW(s, x) \right] dx \\
&= \int_{\Omega} e^{tB} X_0(x) dx \\
&= \sum_{i,j \geq 0}^{\infty} e^{-k_{ij}t} \langle e_{i,j}, X_0 \rangle_H \int_{\Omega} e_{ij}(x) dx, \quad k_{ij} = D\lambda_{i,j} + 0.5. \tag{135}
\end{aligned}$$

We also have

$$\begin{aligned}
\mathbb{E}\Phi_2(X(t)) &= \mathbb{E} \| e^{tB} X_0 + \int_0^t e^{(t-s)B} dW(s) \|^2 \tag{136} \\
&= \mathbb{E} \| e^{tB} X_0 \|^2 + \mathbb{E} \left\| \int_0^t e^{(t-s)B} dW(s) \right\|^2 + 2\mathbb{E} \langle e^{tB} X_0, \int_0^t e^{(t-s)B} dW(s) \rangle_H \tag{137}
\end{aligned}$$

Using the spectral decomposition, the fact that $X_0 \in H$ and the Ito isometry yields

$$\mathbb{E}\Phi_2(X(t)) = \sum_{i,j \geq 0}^{\infty} e^{-2k_{ij}t} \langle e_{i,j}, X_0 \rangle_H^2 + \sum_{i,j \geq 0}^{\infty} \left\langle \frac{q_{ij}}{2k_{ij}} (1 - e^{-2k_{ij} \Delta t}) \right\rangle_H. \tag{138}$$

To compute the exact solutions $\mathbb{E}\Phi_1(X(t))$ and $\mathbb{E}\Phi_2(X(t))$, we can either truncate (135) and (138) by using the first N_h terms, N_h being the number of finite element test basis functions, or use directly the Monte Carlo method with (132).

Our code was implemented in Matlab 8.1. We use two different initial solutions with each test function Φ_1 and Φ_2 . Indeed we use the initial solution $X_0 = X_0^{(2)} = 0$ for Φ_2 , and

$$X_0 = X_0^{(1)} = \sum_{i,j \geq 1}^{\infty} q_{i,j}^1 e_{i,j}, \quad q_{i,j}^1 = (i^2 + j^2)^{-1.001}$$

for Φ_1 . Our finite element triangulation has been done from rectangular grid of maximal length h . Fast Leja Points technique as presented in [30, 9] is used to compute the exponential matrix function $S_h^1 = \varphi_1$. In the legends of our graphs, we use the following notations

- "TimeError1" denotes the weak error for fixed $h = 1/150$ with $\Phi = \Phi_1$, $X_0 = X_0^{(1)}$ and final time $T = 1$.
- "TimeError2" denotes the weak error for fixed $h = 1/50$ with $\Phi = \Phi_2$, $X_0 = X_0^{(2)}$ and final time $T = 1$.
- "SpaceError1" denotes the weak error for fixed time step $\Delta t = 1/500$ with $\Phi = \Phi_1$, $X_0 = X_0^{(1)}$ and final time $T = 0.1$.
- "SpaceError2" denotes the weak error for fixed time step $\Delta t = 1/20000$ with $\Phi = \Phi_2$ and $X_0 = X_0^{(2)}$ and final time $T = 0.1$.

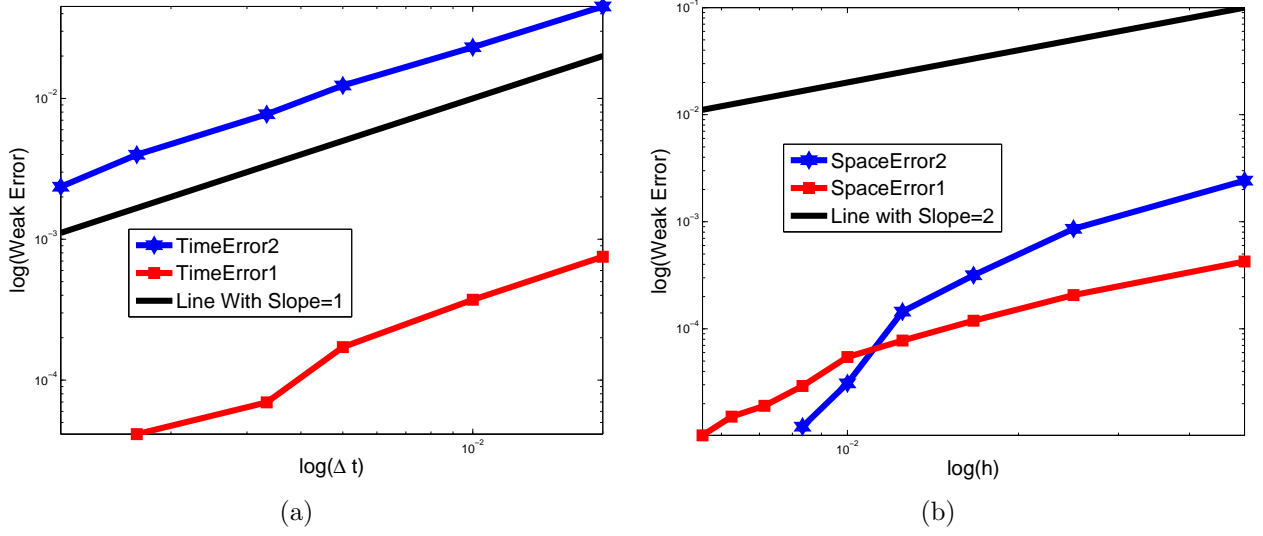


Figure 1: (a) Weak convergence in time of the exponential scheme for $\Phi = \Phi_1$ with $X_0 = X_0^{(1)}$ (TimeError1) and $\Phi = \Phi_2$ with $X_0 = X_0^{(2)}$ (TimeError2). (b) Weak convergence in space of the exponential scheme for $\Phi = \Phi_1$ with $X_0 = X_0^{(1)}$ (SpaceError1), and $\Phi = \Phi_2$ with $X_0 = X_0^{(2)}$ (SpaceError2).

In all our graphs $D = 0.1, \beta = 1, \delta = 0.001$, the weak errors are computed at the final time T and 50 realizations are used to estimate the weak errors with the Monte Carlo method.

Figure 1(a) shows the weak convergence in time of the exponential scheme for $\Phi = \Phi_1$ with $X_0 = X_0^{(1)}$, and $\Phi = \Phi_2$ with $X_0 = X_0^{(2)}$. The weak order of convergence in time is respectively 1.1 for $\Phi = \Phi_1$ and 1.008 for $\Phi = \Phi_2$. These orders of convergence are then closed to the optimal order 1 obtained in Remark 6.3.

For space convergence, very small time steps are needed and the weak errors are performed at small final time $T = 0.1$, Figure 1(b) shows the weak convergence in space of the exponential scheme for $\Phi = \Phi_1$ with $X_0 = X_0^{(1)}$, and $\Phi = \Phi_2$ with $X_0 = X_0^{(2)}$. The weak order of convergence in space is respectively 1.71 for $\Phi = \Phi_1$ and 2.01 for $\Phi = \Phi_2$. These orders of convergence are also closed to the optimal order 2 obtained in Remark 6.3. To sum up, our simulations confirm the theoretical results obtained in Remark 6.3.

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